A Local Field of Extremals for Optimal Control Problems with State Constraints of Relative Degree 1

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Abstract

A local embedding of a boundary arc into a field of extremals is constructed for single-input optimal control problems with state space constraints given by control-invariant manifolds of relative degree 1. The strong local optimality of the reference trajectory is proven.

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1 Introduction

Many practical problems of engineering or scientific interest can be formulated in the framework of optimal control theory with state space constraints. Examples come from various disciplines like the space shuttle re-entry problem considered in [4], the problem of minimizing the base transit time in bipolar transistors [25], or simple models for optimal control problems in cancer chemotherapy [10]. Despite its importance, there still exists a large gap between the theories of necessary and sufficient conditions for optimality for optimal control problems with state space constraints. In fact, even for necessary conditions there are numerous (and in their details not all equivalent) formulations (e.g. [13, 18, 29]). This is due to the presence of measures as multipliers when state constraints are active and, depending on the generality in which these constraints are formulated, these measures in principle can have a part which is singular with respect to Lebesgue measure. On the other hand, in many practical applications state constraints have strong geometric properties, often they are embedded submanifolds, and it should be possible to give strengthened necessary conditions for optimality in the sense of specifying the measures further. In particular, it is of importance to give conditions under which there are no singular parts. There exist formulations of the necessary conditions along these lines in the survey [12], but these results still are stated as “Informal Theorem” with the comment that the result “has not, to our knowledge, been proved fully in the literature” with the notable exception of Maurer’s results [18, 19]. In these papers the structure of multipliers is analyzed assuming that

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trajectories are finite concatenations of arcs which either lie on the constraint or in the interior of the state space. While these results are thus somewhat restrictive, the fact remains that they apply in many practical problems and provide a concrete set of conditions that allow to solve problems numerically and analytically. Thus, while "Informal Theorem 4.1" in [12] has not been established rigorously in all its details, it nevertheless provides a useful basis for sufficient conditions for optimality.

The literature on sufficient conditions for optimality for optimal control problems with state space constraints is limited. There exist general results for sufficiency in terms of existence of viscosity solutions to the corresponding Hamilton-Jacobi-Bellman equation which are applicable to problems with state space constraints (e.g. [7, 27]). However, their practical usefulness is limited by the difficulties of in fact finding these solutions. While this can be done numerically for specific problems, this does not provide any geometric insights into the structure of optimal solutions. Stalford's results in [28] follow a synthesis approach along the ideas of Boltyanskii's "regular synthesis" [3], but with equally restrictive assumptions. A general theory of regular synthesis, like it was developed by Piccoli and Sussmann in [23] for problems without state space constraints, does not exist for problems with state space constraints. However, a significant step towards understanding the structure of optimal solutions for problems with state constraints geometrically has been made by Bonnard et al. in [4] where a local analysis of time optimal syntheses for single input systems with state constraints has been initiated.

Our analysis here follows this geometric approach and we analyze the structure of a local field of extremals around a reference trajectory with boundary arcs (a piece of the trajectory or its graph that lies on the state constraint) if the state constraints are control invariant submanifolds of relative degree 1. In this case, following Maurer's reasoning [19], it can be shown that the measures associated with the state constraints are absolutely continuous with respect to Lebesgue measure, and that strengthened versions of the necessary conditions for optimality guarantee the local embedding of the reference trajectory into a local field of extremals. The local embedding of a boundary arc clarifies the roles of the measures and provides the link between boundary arcs and the trajectories away from the constraint. It is different from the classical local imbeddings for unconstrained problems in the sense that, because of the presence of the constraint, this local field necessarily contains small pieces of trajectories which when propagated backward are not close to the reference trajectory. This, however, does not effect the memoryless properties required for a synthesis forward in time and strong local optimality of the reference trajectory follows from the existence of the field.

In this paper we describe the local synthesis under conditions that in some sense can be considered a typical or common scenario. In fact, we propagate a local synthesis when the value function for the problem is differentiable along the reference trajectory. An outline of the construction of the field for this case has already appeared without proofs in a preliminary version in [16]. Here the construction is carried out in detail and the proof of the optimality of the reference trajectory is given.

2 Mathematical Model

We consider a time-varying single-input system which is linear in the control. The restriction to single-input systems is significant in that it simplifies the problem and excludes complications due to interactions of the controls. But our results can easily be modified to hold for more general than control-linear systems. However, restricting the construction to this class makes the reasoning cleaner and the conditions more natural. We consider the following model formulated as a Meyer Problem:

(Q) minimize

$$J(u) = \varphi(T, x(T)) \tag{1}$$

over all Lebesgue measurable functions u, $u : [t_0, T] \to [a, b]$, with values in a compact interval $[a, b] \subset \mathbb{R}$, subject to the dynamics

$$\dot{x}(t) = f(t, x) + ug(t, x), \quad x(t_0) = x_0, \tag{2}$$

terminal constraints

$$\psi(T, x(T)) = 0, \tag{3}$$

and state space constraints

$$h_\alpha(t, x) \leq 0 \quad \text{for } \alpha = 1, \dots, r. \tag{4}$$

Assumptions: The time-varying vector fields f and g, $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, are continuously differentiable in (t, x). The terminal constraint $N = \{(t, x) : \psi(t, x) = 0\}$ is defined by a continuously differentiable function $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^k$ and we assume that the gradients $\nabla \psi_i = (\frac{\partial \psi_i}{\partial t}, \frac{\partial \psi_i}{\partial x}) \in (\mathbb{R}^{n+1})^*$ are linearly independent on N. In this formulation the terminal time T is considered free; a fixed terminal time simply would be described by one of the functions ψ_i. Similarly, the state-space constraints $M_\alpha = \{(t, x) : h_\alpha(t, x) = 0\}$, $\alpha = 1, \dots, r$, are defined by time-varying continuously differentiable vector fields $h_\alpha : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $(t, x) \mapsto h_\alpha(t, x)$. However, now we assume that the gradients $\nabla h_\alpha = (\frac{\partial h_\alpha}{\partial t}, \frac{\partial h_\alpha}{\partial x}) \in (\mathbb{R}^{n+1})^*$ are once more continuously differentiable. Like for ψ, we assume that the gradients ∇h_α do not vanish on M_α. In particular, each M_α thus is an embedded submanifold of codimension 1 of $\mathbb{R}^{n+1}$.

Problems with an objective of the form

$$J(u) = \int_{t_0}^{T} (L_0(t, x) + uL_1(t, x))\, dt + \varphi(T, x(T)) \tag{5}$$

where the Lagrangian is linear in the control easily convert into problem (Q) at the expense of adding one extra state variable x_{n+1} with dynamical equation

$$\dot{x}_{n+1} = L_0(t, x) + uL_1(t, x), \quad x_{n+1}(t_0) = 0, \tag{6}$$

and redefining the penalty function as $\phi(t, x, x_{n+1}) = x_{n+1} + \varphi(t, x)$. In fact, this is the structure for the problem of minimizing the base transit time in semiconductor devices [25, 30], or for mathematical models for cancer chemotherapy [10]. Both these problems are described by a time-invariant, single-input control system over a fixed finite interval $[0, T]$ that corresponds to the length of the base in the semiconductor problem [25] or a predetermined therapy interval in the chemotherapy models [10]. In both cases, although the systems are time-invariant, because of the fixed horizon, the optimal controls become "time"-dependent, $u = u(t, x)$, and the synthesis needs to be constructed on $\mathbb{R} \times \mathbb{R}^n$. (Optimal controls in both cases do not only depend on the state, but the doping profile and drug schedule also depend on how close the system is to the terminal time.) It then makes conceptually no difference to allow for time-dependence in the dynamics and we prefer to formulate the problem as such.

Clearly, problem (Q) also could be formulated as a time-invariant system adding the trivial dynamics $\dot{x}_0 = 1$ with initial condition $x_0(t_0) = t_0$. (This also is in agreement with assuming that the time-varying vector fields f and g are continuously differentiable in t.) However, in view of the given application oriented problems which motivate this research, we prefer to keep time separately as a variable since there is a definite distinction between the variable t and the variable x in these problems. In this way our results are more directly transferable to these applications. The only change this necessitates is that in the flow for the system we have to consider the *graphs of trajectories*, rather than trajectories themselves.

For this we want to fix our notation: Given a control $u : [t_0, T] \to [a, b]$, the initial value problem (2) has a unique solution defined on some maximal open interval of definition I. For a control to be admissible we require that $I \supset [t_0, T]$ and we call the solution the *trajectory* corresponding to the control u; the pair (x, u) is a *controlled trajectory*. Given an open subset P of $\mathbb{R}^n$, let t_{in} and t_f be two continuous functions satisfying $t_{in}(p) < t_f(p)$ for all $p \in P$ and let $D = \{(t, p) : t_{in}(p) \le t \le t_f(p), p \in P\}$. If $u = u(t, p)$ denotes some parameterized family of admissible controls defined on D, let $x = x(t, p)$ denote the corresponding trajectories. Then we define the corresponding flow as the map given by the graphs of the trajectories, i.e.

$$\sigma : D \to \mathbb{R} \times \mathbb{R}^n, \ (t, p) \mapsto (t, x(t, p)). \tag{7}$$

If the time-varying equations are written as an autonomous system, this exactly is the standard flow of the trajectories. Similarly, if we set

$$F(t, x) = \begin{pmatrix} 1 \\ f(t, x) \end{pmatrix} \quad \text{and} \quad G(t, x) = \begin{pmatrix} 0 \\ g(t, x) \end{pmatrix}, \tag{8}$$

then for a continuously differentiable function $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, the functions

$$\mathcal{L}_F k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x) \mapsto (\mathcal{L}_F k)(t, x) = \frac{\partial k}{\partial t}(t, x) + \frac{\partial k}{\partial x}(t, x) f(t, x)$$

and

$$\mathcal{L}_G k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x) \mapsto (\mathcal{L}_G k)(t, x) = \frac{\partial k}{\partial x}(t, x) g(t, x)$$

are the Lie-derivatives of the function k along the vector fields F and G, respectively. In terms of this notation, the derivative of the function h_α (defining the manifold M_α) along trajectories of the system is given by

$$\dot{h}_\alpha(t, x(t)) = \frac{d}{dt} h_\alpha(t, x(t)) = \mathcal{L}_F h_\alpha(t, x(t)) + u(t) \mathcal{L}_G h_\alpha(t, x(t)).$$

If the function $\mathcal{L}_G h_\alpha$ does not vanish at a point $(\tilde{t}, \tilde{x}) \in M_\alpha$, then there exists a neighborhood V of $(\tilde{t}, \tilde{x})$ such that there exists a unique control $u_\alpha = u_\alpha(t, x)$ which solves the equation $\dot{h}_\alpha(t, x) = 0$ on V and u_α is given in feedback form as

$$u_\alpha(t, x) = -\frac{\mathcal{L}_F h_\alpha(t, x)}{\mathcal{L}_G h_\alpha(t, x)}. \tag{9}$$

Definition 2.1 *We call the manifold M_α* control-invariant *of* relative degree 1 *for problem* (Q) *if the Lie derivative of h_α with respect to G does not vanish anywhere on M_α and if the function $u_\alpha(t,x)$ defined by (9) is admissible, i.e. takes values in the control set $[a,b]$.*

Since we are dealing with a problem with control constraints, here we explicitly include the requirement that the control u_α satisfies these constraints in the definition. Thus, for a control-invariant submanifold of relative degree 1, the control that keeps the manifold invariant is unique and the corresponding dynamics (2) induces a unique flow on the constraint.

In this paper we assume throughout that:

(A) All constraint manifolds M_α are control-invariant of relative degree 1.

This assumption corresponds to the least degenerate, or, equivalently, most common scenario and is satisfied for many practical problems like, for example, [4, 10, 25].

3 Necessary Conditions for Optimality

3.1 The Maximum Principle with State Constraints

First-order necessary conditions for optimality are given by the *Pontryagin maximum principle* [22]. Mathematically the presence of the state-space constraints complicates matters in that it brings in additional multipliers which a priori are only known to be non-negative Radon measures. Here we follow [8, 13, 29]: Suppose $u_*:[t_0,T]\to[a,b]$ is an optimal control with corresponding trajectory x_*. For simplicity of presentation we also assume that no state constraints are active at the terminal time so that the standard transversality conditions apply. Then the **necessary conditions for optimality** of u_* state that there exist a constant $\lambda_0\geq 0$, an absolutely continuous function η, which we write as row-vector, $\eta:[t_0,T]\to(\mathbb{R}^n)^*$, and non-negative Radon measures $\mu_\alpha\in C^*([t_0,T];\mathbb{R})$, $\alpha=1,\ldots,r$, with support in the sets $R_\alpha=\{t\in[t_0,T]:\ h_\alpha(t,x_*(t))=0\}$, which do not all vanish simultaneously (in the sense that $\lambda_0+\|\eta\|_\infty+\sum_{\alpha=1}^r\mu_\alpha([t_0,T])>0$) such that with

$$\lambda(t)=\eta(t)-\sum_{\alpha=1}^r\int_{[t_0,t)}\frac{\partial h_\alpha}{\partial x}(s,x_*(s))d\mu_\alpha(s), \tag{10}$$

and

$$H=H(t,\lambda,x,u)=\lambda\left(f(t,x)+ug(t,x)\right) \tag{11}$$

the following conditions hold:

(a) The optimal control minimizes the Hamiltonian over the control set $[a,b]$ along $(\lambda(t),x_*(t))$:

$$H(t,\lambda(t),x_*(t),u_*(t))=\min_{a\leq w\leq b}H(t,\lambda(t),x_*(t),w). \tag{12}$$

(b) The adjoint equation holds in the form

$$\dot\eta(t)=-\lambda(t)\left(\frac{\partial f}{\partial x}(t,x_*(t))+u_*(t)\frac{\partial g}{\partial x}(t,x_*(t))\right), \tag{13}$$

and there exists a row-vector $\mu \in (\mathbb{R}^k)^*$ such that

$$\lambda(T) = \lambda_0 \frac{\partial \varphi}{\partial x}(T, x_*(T)) + \mu \frac{\partial \psi}{\partial x}(T, x_*(T)) \tag{14}$$

and

$$H(T, \lambda(T), x_*(T), u_*(T)) + \lambda_0 \frac{\partial \varphi}{\partial t}(T, x_*(T)) + \mu \frac{\partial \psi}{\partial t}(T, x_*(T)) = 0. \tag{15}$$

We call control-trajectory pairs (x, u) for which there exist multipliers such that these conditions are satisfied *extremals*. In general, it cannot be excluded that λ_0 vanishes and extremals with $\lambda_0 = 0$ are called abnormal, while those with $\lambda_0 > 0$ are called normal. There exist several results which can be used to establish the normality of extremals. For example, see [5, 17, 24] where problems with state space constraints are considered. In our construction of a local embedding below we will also need to assume that:

(B) The reference extremal is normal.

In this case we can normalize $\lambda_0 = 1$ as we do henceforth.

3.2 Regularity of the multipliers for problem (Q)

Since admissible controls are only Lebesgue-measurable, in principle, the sets R_α when a certain constraint is active can be arbitrarily complicated sets. Nevertheless, in most practical situations this set often is a union of intervals and in this case more stringent necessary conditions for optimality are valid which we will use to formulate sufficient conditions for local optimality. Following the notation introduced by Maurer [18], we call a piece Γ of the graph of a trajectory defined over an open interval I which does not meet the boundary of the admissible state space an *interior arc* and call Γ a *boundary arc* if at least one constraint is active on all of I. Since we only consider a single-input control system, typically only one of the constraints will be active, but like in [25] there may be more than one constraint which become active at some time over the interval $[t_0, T]$. More specifically, we call Γ an M_α-boundary arc over I if only the constraint $h_\alpha \leq 0$ is active on I. The times τ when interior arcs and boundary arcs meet are called *junction times* and the corresponding pairs $(\tau, x(\tau))$ *junction points*. We will not consider more complicated scenarios in this paper.

Optimal controls are characterized by the minimum condition (12). Given an extremal (x_*, u_*) we define the corresponding switching function Φ_* as

$$\Phi_*(t) = \lambda(t) g(t, x_*(t)). \tag{16}$$

Then, since the Hamiltonian is linear in u, optimal controls satisfy

$$u_*(t) = \begin{cases} b & \text{if } \Phi_*(t) < 0 \\ a & \text{if } \Phi_*(t) > 0 \end{cases} \tag{17}$$

and we call the constant controls $u = a$ and $u = b$ *bang* controls. A priori the control is not determined by the minimum condition at times when $\Phi_*(t) = 0$. However, if $\Phi_*(t)$ vanishes on an open interval I, then also all its derivatives must vanish and this may determine the control. If the trajectory is an interior arc over I, this leads to the concept of *singular arcs* and well-known results about its optimality, which, however, are not needed in this paper. Here we are interested in the case of boundary arcs. The Propositions below are based on Maurer's arguments and results [18, 19]. For sake of completeness we include a precise formulation as we need it since these properties are central to the construction of a field in section 4.

Proposition 3.1 *Let* Γ_α *be an* M_α*-boundary arc defined over an open interval* $I = (\tau_1, \tau_2)$ *and suppose the corresponding boundary control* u_α *defined by (9) takes values in the interior of the control set along* Γ_α*. Then the Radon measure* μ_α *is absolutely continuous with respect to Lebesgue measure on* I *with continuous and non-negative Radon-Nikodym derivative* $\nu_\alpha(t)$ *given by*

$$\nu_\alpha(t) = \frac{1}{\mathcal{L}_G h_\alpha(t, x_*(t))} \lambda(t) \left(\frac{\partial g}{\partial t}(t, x_*(t)) + [f, g](t, x_*(t)) \right) \tag{18}$$

where $[f, g]$ *denotes the Lie bracket of the time-varying vector fields* f *and* g *in the variable* x*.*

We use the convenient notation

$$[f, g](t, x) = \frac{\partial g}{\partial x}(t, x) f(t, x) - \frac{\partial f}{\partial x}(t, x) g(t, x). \tag{19}$$

In terms of the vector fields F and G on the extended state-space in (t, x) defined in (8), the term $\frac{\partial g}{\partial t} + [f, g]$ is the x-component of the Lie-bracket of F and G; specifically

$$[F, G]_{(t,x)} = \begin{pmatrix} 0 \\ \frac{\partial g}{\partial t} + [f, g] \end{pmatrix}. \tag{20}$$

Proof. Since the time-varying vector field g is continuously differentiable, it follows that the composition $t \mapsto g(t, x_*(t))$ is absolutely continuous on I. Since the boundary control takes values in the interior of the control set, it follows from the minimum condition (12) that $\Phi_*(t) = \lambda(t) g(t, x_*(t)) \equiv 0$. This implies that λ is continuous on I. For, by (10), λ is continuous from the left and at every time $\tau \in I$ we have both $\lambda(\tau) g(\tau, x_*(\tau)) = 0$ and $\lambda(\tau+) g(\tau, x_*(\tau)) = 0$ (otherwise there would exist times $t > \tau$ violating the minimum condition). But if follows from (10) that there exists a constant $\nu \geq 0$ such that $\lambda(\tau+) = \lambda(\tau) - \frac{\partial h_\alpha}{\partial x}(\tau, x_*(\tau))\nu$ and thus

$$\nu \frac{\partial h_\alpha}{\partial x}(\tau, x_*(\tau)) g(\tau, x_*(\tau)) = \nu \mathcal{L}_G h_\alpha(\tau, x_*(\tau)) = 0$$

implying $\nu = 0$ since $\mathcal{L}_G h_\alpha \neq 0$. In particular, the measure μ_α does not have any atomic parts on I.

In fact, the measure μ_α is absolutely continuous with respect to Lebesgue measure on I. Since $\Phi_*(t) = \lambda(t) g(t, x_*(t)) \equiv 0$, by (10) the quantity

$$\left(\int_{[t_0, t)} \frac{\partial h_\alpha}{\partial x}(s, x_*(s)) d\mu_\alpha(s) \right) g(t, x_*(t))$$

is absolutely continuous on I. The measure $\mu_\alpha \in C^*([t_0, T]; \mathbb{R})$ can be represented in a unique way by a function $F_\alpha : [t_0, T] \to \mathbb{R}$ of normalized bounded variation (i.e. $F_\alpha(t_0) = 0$) such that for any continuous function $\phi : [t_0, T] \to \mathbb{R}$ we have $\langle \mu_\alpha, \phi \rangle = \int_{[t_0, T]} \phi(s) dF_\alpha(s)$ where the integral is the Lebesgue-Stieltjes integral. Since μ_α is non-negative, the function F_α is monotonically increasing. Writing $\mu_\alpha(t) = \int_{[t_0, t)} dF_\alpha(s)$, it therefore follows from integration by parts that

$$\int_{[t_0, t)} \frac{\partial h_\alpha}{\partial x}(s, x_*(s)) d\mu_\alpha(s) = \frac{\partial h_\alpha}{\partial x}(t, x_*(t)) \mu_\alpha(t) - \int_{[t_0, t)} \left[F_\alpha(s) \frac{d}{ds} \left(\frac{\partial h_\alpha}{\partial x}(s, x_*(s)) \right) \right] ds.$$

Hence

$$\left(\frac{\partial h_\alpha}{\partial x}(t,x_*(t))g(t,x_*(t))\right)\mu_\alpha(t)=\mathcal{L}_G h_\alpha(t,x_*(t))\mu_\alpha(t)$$

is absolutely continuous with respect to Lebesgue measure. Since the constraint M_α has relative degree 1, the term in parenthesis does not vanish and by our assumptions on h_α and g, it is also absolutely continuous. Thus $\mu_\alpha(t)$ itself is absolutely continuous. If we denote the Radon-Nikodym derivative by ν_α, then the non-negativity of ν_α follows from the fact that μ_α is non-negative. An explicit formula for the Radon-Nikodym derivative can now be derived by differentiating the switching function. A direct computation verifies that

$$\begin{aligned}0=\dot\Phi_*(t)&=\lambda\left(\frac{\partial g}{\partial t}(t,x_*)+\frac{\partial g}{\partial x}(t,x_*)\left(f(t,x_*)+ug(t,x_*)\right)\right)\\&-\left(\lambda\left(\frac{\partial f}{\partial x}(t,x_*)+u\frac{\partial g}{\partial x}(t,x_*)\right)+\nu_\alpha\frac{\partial h_\alpha}{\partial x}(t,x_*)\right)g(t,x_*)\\&=\lambda\left(\frac{\partial g}{\partial t}(t,x_*)+[f,g](t,x_*)\right)-\nu_\alpha(t)\mathcal{L}_G h_\alpha(t,x_*).\end{aligned}\tag{21}$$

Again, since $\mathcal{L}_g h_\alpha$ doesn't vanish on M_α, equation (21) gives (18). □

In particular, if $I=(\tau_1,\tau_2)$ is an M_α-boundary arc, then the adjoint equation can be rewritten in the form

$$\dot\lambda(t)=-\lambda(t)\left(\frac{\partial f}{\partial x}(t,x_*)+u_*\frac{\partial g}{\partial x}(t,x_*)\right)-\nu_\alpha(t)\frac{\partial h_\alpha}{\partial x}(t,x_*),\tag{22}$$

where all partial derivatives are evaluated along the reference trajectory. The following continuity condition on the multiplier at junction times directly follows from Maurer's junction conditions [18] (c.f. also, [12, Prop. 4.2] or [4, Lemma 2.4]). For completeness sake the proof is included.

Proposition 3.2 *Let τ be an entry or exit junction time between an interior arc and an M_α-boundary arc and suppose the reference control u_* has a limit at τ along the interior arc. Then the interior arc is transversal to M_α at entry or exit if and only if the control u_* is discontinuous at τ. In this case the multiplier λ remains continuous at τ.*

Proof. Let $\bar u$ be the limit of the reference control along the interior arc at time τ. The interior arc is transversal to M_α if and only if

$$\frac{d}{dt}h_\alpha(t,x_*(t))_{|t=\tau}=\mathcal{L}_F h_\alpha(\tau,x_*(\tau))+\bar u\mathcal{L}_G h_\alpha(\tau,x_*(\tau))\neq 0.\tag{23}$$

But along the boundary control u_α we have

$$\mathcal{L}_F h_\alpha(\tau,x_*(\tau))+u_\alpha(\tau,x_*(\tau))\mathcal{L}_G h_\alpha(\tau,x_*(\tau))=0\tag{24}$$

and, since M_α is of relative degree 1, the Lie derivative $\mathcal{L}_G h_\alpha(\tau,x_*(\tau)$ does not vanish. Hence (23) can only vanish if $\bar u=u_\alpha(\tau,x_*(\tau))$. In particular, an interior arc is transversal to M_α at entry or exit times if and only if the control is discontinuous at τ.

We now assume that the reference control u_* is discontinuous at τ and show that λ remains continuous at τ. In principle the measure μ_α could have an atomic part at $t=\tau$ which would generate a jump in λ at time τ. But by (10) it follows that there exists a constant $\nu \geq 0$ so that

$$\lambda(\tau-) = \lambda(\tau+) + \nu \frac{\partial h_\alpha}{\partial x}(\tau, x_*(\tau)). \tag{25}$$

Hence at both entry and exit junctions

$$\Phi_*(\tau-) = \Phi_*(\tau+) + \nu \mathcal{L}_G h_\alpha(\tau, x_*(\tau)).$$

If τ is an entry time, then by the continuity of the switching function on I we have $\Phi_*(\tau+) = 0$. Suppose $\Phi_*(\tau-) \neq 0$ and first consider the case $\Phi_*(\tau-) > 0$. In this case the control is constant over some interval $(\tau - \varepsilon, \tau)$ and given by a. Since the interior arc is transversal to M_α at $(\tau, x_*(\tau))$, we have

$$\mathcal{L}_F h_\alpha(\tau, x_*(\tau)) + a\mathcal{L}_G h_\alpha(\tau, x_*(\tau)) > 0$$

and thus from (24) we obtain

$$(a - u_\alpha(\tau, x_*(\tau)))\, \mathcal{L}_G h_\alpha(\tau, x_*(\tau)) > 0.$$

But $a < u_\alpha(\tau, x_*(\tau))$ and so this implies $\mathcal{L}_g h_\alpha(\tau, x_*(\tau)) < 0$ giving the contradiction

$$0 < \Phi_*(\tau-) = \nu \mathcal{L}_G h_\alpha(\tau, x_*(\tau)) \leq 0.$$

Similarly, if $\Phi_*(\tau-) < 0$, then the control is given by b over some interval $(\tau - \varepsilon, \tau)$, but also

$$\mathcal{L}_F h_\alpha(\tau, x_*(\tau)) + b\mathcal{L}_G h_\alpha(\tau, x_*(\tau)) > 0.$$

In this case,

$$(b - u_\alpha(\tau, x_*(\tau)))\, \mathcal{L}_G h_\alpha(\tau, x_*(\tau)) > 0$$

and $b > u_\alpha(\tau, x_*(\tau))$ implies $\mathcal{L}_g h_\alpha(\tau, x_*(\tau)) > 0$ giving the contradiction

$$0 > \Phi_*(\tau-) = \nu \mathcal{L}_G h_\alpha(\tau, x_*(\tau)) \geq 0.$$

Thus $\Phi(\tau-) = 0$, i.e. the switching function remains continuous at τ. But then, as in the proof of Proposition 3.1, equation (25) in connection with an order 1 state space constraint implies that

$$\nu \frac{\partial h_\alpha}{\partial x}(\tau, x_*(\tau)) g(\tau, x_*(\tau)) = \nu \mathcal{L}_G h_\alpha(\tau, x_*(\tau)) = 0$$

giving $\nu = 0$. With the obvious modification this argument also applies to exit junctions. Thus the switching function and the multiplier remain continuous at entry and exit junctions. □

We summarize the form of the necessary condition for problem (Q). These conditions agree with "INFORMAL THEOREM 4.1" in [12].

Theorem 3.1 *Suppose the state space constraints in problem* (Q) *are given by control-invariant submanifolds of relative degree* 1*. Let* (x_*, u_*) *be a normal reference extremal defined over the interval* $[t_0, T]$ *with the property that the graph of* x_* *is a finite concatenation of interior and boundary arcs with junction times* t_i^*, $i = 1, \ldots, m$, $t_0 = t_0^* < t_1^* < \ldots < t_m^* < t_{m+1}^* = T$*. Suppose on each interval* (t_i^*, t_{i+1}^*), $i = 0, \ldots, m$, *at most one constraint is active and the boundary* *controls take values in the interior of the control set and the interior controls are given by bang controls. Also assume no state-space constraint is active at the terminal time. Then there exists an absolutely continuous function* λ_*, $\lambda_* : [t_0, T] \to (\mathbb{R}^n)^*$, *with*

$$\lambda_*(T) = \frac{\partial \varphi}{\partial x}(T, x_*(T)) + \mu \frac{\partial \psi}{\partial x}(T, x_*(T))$$

for some $\mu \in (\mathbb{R}^k)^*$, *such that*

(1) *On each interval* (t_i^*, t_{i+1}^*), $i = 0, \dots, m$, λ_* *is continuously differentiable and satisfies the adjoint equation in the form*

$$\dot{\lambda}_*(t) = -\lambda_*(t) \left(\frac{\partial f}{\partial x}(t, x_*) + u_* \frac{\partial g}{\partial x}(t, x_*) \right) - \sum_{\alpha=1}^{r} \nu_\alpha(t) \frac{\partial h_\alpha}{\partial x}(t, x_*),$$

where

$$\nu_\alpha(t) = \frac{1}{\mathcal{L}_G h_\alpha(t, x_*(t))} \lambda_*(t) \left(\frac{\partial g}{\partial t}(t, x_*(t)) + [f, g](t, x_*(t)) \right)$$

if (t_i^*, t_{i+1}^*) *is the domain of an* M_α*-boundary arc and* $\nu_\alpha(t) = 0$ *if the constraint* M_α *is not active.*

(2) *With* $\Phi_*(t) = \lambda_*(t) g(t, x_*(t))$ *the control satisfies*

$$u_*(t) = \begin{cases} b & \text{if } \Phi_*(t) < 0 \\ a & \text{if } \Phi_*(t) > 0 \end{cases}.$$

Along an M_α*-boundary arc the control* u_* *is given by*

$$u_*(t) = -\frac{\mathcal{L}_F h_\alpha(t, x_*(t))}{\mathcal{L}_G h_\alpha(t, x_*(t))}.$$

4 Construction of a Local Field of Extremals Near a Boundary Trajectory

Our aim is to formulate sufficient conditions for strong local optimality of a reference extremal (x_*, u_*) which is a finite concatenation of interior arcs and M_α-boundary arcs. These results are geometric in nature constructing a local field of extremals around the reference trajectory. However, even under codimension 0 conditions, there does not exist a unique structure this field would have. It obviously matters whether interior arcs correspond to singular or bang controls. A local synthesis for the least degenerate junctions between boundary arcs and interior singular arcs is described in [26]. Here we only consider junctions of boundary arcs with interior bang arcs. But even in this case, nonequivalent syntheses are possible. For example, it matters whether there are terminal constraints in the model (also see the conclusion). In this paper we analyze the case when the *interior trajectories are imbedded into a field of bang extremals* corresponding to a constant control. For fixed-time problems this situation is the typical one if no terminal constraints are imposed, like, for example, for the chemotherapy problems considered in [10]. The new aspect of our construction is the embedding of a boundary arc.

4.1 Assumptions on the reference trajectory

We assume the following scenario:

(C1) Let Γ_α be an M_α-boundary arc of an extremal input-trajectory pair $\Gamma = (x_*, u_*)$ defined over an interval $[\tau_1, \tau_2] \subset (t_0, T)$, $\tau_1 < \tau_2$, with corresponding multipliers λ_* and ν_α. Suppose τ_1 and τ_2 are the entry- and exit-times, respectively, and assume there exists an $\varepsilon > 0$ such that the switching function Φ_* is positive on $[\tau_1 - \varepsilon, \tau_1)$ and negative on $(\tau_2, \tau_2 + \varepsilon]$.

Hence the control u_* is given by $u_*(t) \equiv a$ on $(\tau_1 - \varepsilon, \tau_1)$ and $u_*(t) \equiv b$ on $(\tau_2, \tau_2 + \varepsilon)$. On the interval (τ_1, τ_2), the control is given by the feedback control that leaves M_α invariant, i.e. $u_*(t) = u_\alpha(t, x_*(t))$. We denote the entry- and exit-arcs by Γ_- and Γ_+.

(C2) Along the boundary arc the control $u_*(t) = u_\alpha(t, x_*(t))$ takes values in the interior of the control set for all times t in the closed interval $[\tau_1, \tau_2]$.

(C3) The multiplier $\nu_\alpha(t)$ is positive on $[\tau_1, \tau_2]$.

The choice of a as entry- and b as exit-control is without loss of generality, but it puts some normalization on the structure of the problem. It follows from $(C2)$ that the switching function Φ_* vanishes identically over $[\tau_1, \tau_2]$. Since the control u_* is discontinuous at entry and exit, it follows from Proposition 3.2 that the multiplier λ_*, and thus also the switching function Φ_*, are continuous at τ_1 and τ_2. The derivative of the switching function along an interior arc is given by

$$\dot{\Phi}_*(t) = \lambda_*(t) \left(\frac{\partial g}{\partial t}(t, x_*(t)) + [f, g](t, x_*(t)) \right) \tag{26}$$

and thus, by (21), we therefore have for the entry- and exit-junctions that

$$\dot{\Phi}_*(\tau_1-) - \nu_\alpha(\tau_1) \mathcal{L}_G h_\alpha(\tau_1, x_*(\tau_1)) = \dot{\Phi}_*(\tau_1+) = 0$$

and

$$0 = \dot{\Phi}_*(\tau_2-) = \dot{\Phi}_*(\tau_2+) - \nu_\alpha(\tau_2) \mathcal{L}_G h_\alpha(\tau_2, x_*(\tau_2)).$$

By $(C3)$ the multiplier ν_α is positive and $\mathcal{L}_G h_\alpha$ does not vanish since M_α is control-invariant of degree 1. Our normalization in $(C1)$ thus implies that

$$\dot{\Phi}_*(\tau_1-) < 0, \quad \dot{\Phi}_*(\tau_2+) < 0, \quad \text{and} \quad \mathcal{L}_G h_\alpha < 0. \tag{27}$$

We also need to make assumptions to the effect that we can embed the terminal portion of the reference extremal defined over the interval $[\tau_2, T]$ into a local field of extremals. Clearly, if this portion already is not optimal, there is no need to propagate the embedding further. Recall that the value function at a point (t, x) is defined as the infimum over the values of the objective for all admissible controls when x is taken as the initial condition at initial time t. However, since our construction is only local, it suffices to consider the following local analogue V^* of the value function,

$$V^*(t, x) = \inf_{u \in \mathcal{W}} \varphi(T, x(T)), \tag{28}$$

where $\mathcal{W}$ denotes the set of all admissible controls for problem (Q) with the property that the trajectories with initial condition x at time t have graphs that lie in some fixed neighborhood

W of the graph of the reference trajectory. Such a function V^* can be constructed through the method of characteristics (adapted to the optimal control framework). Assuming that no state constraint is active at the terminal time, for a problem of type (Q) a local embedding of the reference trajectory Γ typically can be constructed by varying the terminal point for the state x on the terminal manifold and the terminal value of the multiplier λ according to the transversality condition (14). If no state space constraints are active along the reference trajectory over this interval, and if the corresponding family covers the state-space injectively with some regularity assumptions related to "least degenerate" conditions satisfied, then the existence of such an embedding which defines V^* and associated sufficient conditions for strong optimality follow from the corresponding results for unconstrained problems. There exist many classical results in this direction (e.g. [2, 6, 11]), but there also has been strong renewed interest in this topic recently, (e.g. [1, 20, 21]). A presentation using geometric constructions similar to this paper is given in our paper [21]. (If state constraints are active along the terminal segment, then the construction to be done here needs to be invoked, but for the moment we may simply assume the boundary arc Γ_α is the last boundary arc along the reference trajectory.)

The local embedding of the boundary arc Γ_α must be a propagation of a local embedding defined over an interval $[\tau_2+\varepsilon, T]$ for some small enough $\varepsilon > 0$ and thus naturally depends on the structure of the local field over the terminal portion $[\tau_2+\varepsilon, T]$. Here we describe in detail one particular codimension 0 scenario that arises when the exit arc Γ_+ is embedded into a family of extremal bang arcs all corresponding to the control $u \equiv b$ and the value function V^* associated to the problem (Q) is differentiable along the reference trajectory. This scenario is the typical one, for example, in the mathematical models for chemotherapy of the type considered in [14, 15] if state-space constraints are imposed. But obviously, the value function need not be differentiable, and clearly the underlying problem matters. Another common scenario arises, for example, in the textbook problem of time-optimal control to the origin for the double integrator if an upper and lower bound on the speed is imposed. This type of local synthesis also arises in low-dimensional problems when the exit arc is singular and is outlined in [26]. In order to limit the length of this paper and be able to give complete proofs, we only carry out the construction of the synthesis for the first situation, but we will comment on the other case below in the conclusion.

Thus we henceforth more specifically also assume:

(D) There exist a sufficiently small $\varepsilon > 0$ and an open neighborhood P of $p_* = x_*(\tau_2+\varepsilon)$ so that *(i)* condition (C) is satisfied, *(ii)* for $t = \tau_2+\varepsilon$ the value function V^* of the problem (Q) is twice continuously differentiable in x and *(iii)* the multiplier λ_* associated with the reference trajectory satisfies

$$\lambda_*(\tau_2+\varepsilon) = \frac{\partial V^*}{\partial x}(\tau_2+\varepsilon, x_*(\tau_2+\varepsilon)). \tag{29}$$

If the reference extremal is in fact optimal, and if the value function is differentiable at $(\tau_2+\varepsilon, x_*(\tau_2+\varepsilon))$, then relation (29) must be satisfied. Conversely, this relation allows to embed the exit arc Γ_+ into a local family of bang extremals corresponding to the control $u = b$ by specifying the values for the adjoint variables at time $\tau_2+\varepsilon$. We now carry out the construction of this local field.

4.2 Construction of a C^1-parameterized family $\mathcal{E}$ of extremals Γ_p near the exit-junction for the problem without state space constraint

We start with constructing a local embedding of the reference trajectory over some interval $[\tau_2 - \delta, \tau_2 + \varepsilon]$, $\delta > 0$, for the problem *without state constraints.* Throughout the construction, whenever necessary, we tacitly shrink P to ensure that the statements made are valid. Since the switching function Φ_* is negative on $(\tau_2, \tau_2 + \varepsilon]$, for $(t, p) \in [\tau_2 - \delta, \tau_2 + \varepsilon] \times P$ define a control u_+ as $u_+(t, p) \equiv b$ and let $x_+(t, p)$ be the solution to the corresponding dynamics,

$$\dot{x} = f(t, x) + bg(t, x), \tag{30}$$

with terminal condition

$$x_+(\tau_2 + \varepsilon, p) = p, \qquad p \in P. \tag{31}$$

For δ and P small enough these solutions exist on all of $[\tau_2 - \delta, \tau_2 + \varepsilon]$ and x_+ is continuously differentiable in both variables. We want the pair (x_+, u_+) to be an extremal (for the problem without state space constraint) and for this we also need to integrate the adjoint equation. Let $\lambda_+(t, p)$ denote the solution to the adjoint equation

$$\dot{\lambda} = -\lambda \left(\frac{\partial f}{\partial x}(t, x_+(t, p)) + b\frac{\partial g}{\partial x}(t, x_+(t, p)) \right) \tag{32}$$

with terminal value

$$\lambda_+(\tau_2 + \varepsilon, p) = \frac{\partial V^*}{\partial x}(\tau_2 + \varepsilon, p). \tag{33}$$

This choice simply enforces the correct terminal values for the multipliers at time $\tau_2 + \varepsilon$. It guarantees that the local embedding to be constructed for times $t \leq \tau_2 + \varepsilon$ will match with a given embedding for times $t \geq \tau_2 + \varepsilon$. Since $V^* \in C^2(P)$, λ_+ is continuously differentiable on P.

For (x_+, u_+) to be extremal, the minimum condition (12) needs to be verified. For $(t, p) \in [\tau_2 - \delta, \tau_2 + \varepsilon] \times P$ define the associated switching functions Φ_+ by

$$\Phi_+(t, p) = \lambda_+(t, p) g(t, x_+(t, p)). \tag{34}$$

Then $\Phi_+(t, p_*)$ agrees with the switching function $\Phi_*(t)$ for the reference trajectory on $[\tau_2, \tau_2 + \varepsilon]$ and thus we have $\Phi_+(t, p_*) < 0$ on $(\tau_2, \tau_2 + \varepsilon]$, $\Phi_+(\tau_2, p_*) = 0$, and $\frac{\partial \Phi_+}{\partial t}(\tau_2, p_*) < 0$. By the implicit function theorem the equation $\Phi_+(t, p) = 0$ therefore has a unique solution near (τ_2, p_*) given by a differentiable function $\bar{\tau} = \bar{\tau}(p)$. Without loss of generality we assume this function is defined on P and that $\frac{\partial \Phi_+}{\partial t}(\bar{\tau}(p), p) < 0$ for $p \in P$. Thus, in order to satisfy the minimum condition, the control needs to switch at $t = \bar{\tau}(p)$ from $u_- = a$ to $u_+ = b$. We denote the corresponding switching surface by $\mathcal{S}$,

$$\mathcal{S} = \{(t, x) : t = \bar{\tau}(p),\ x = x_+(\bar{\tau}(p), p),\ p \in P\}.$$

For $t < \bar{\tau}(p)$ define a new control u_- by $u_-(t, p) \equiv a$ and let x_- and λ_- denote the corresponding trajectory and solution to the adjoint equation with terminal conditions $x_+(\bar{\tau}(p), p)$ and $\lambda_+(\bar{\tau}(p), p)$. Again, for δ and P small enough these solutions exist on $[\tau_2 - \delta, \bar{\tau}(p)]$. Define the concatenated controls $u(t, p)$, $(t, p) \in [\tau_2 - \delta, \tau_2 + \varepsilon] \times P$, by a for $\tau_2 - \varepsilon \leq t \leq \bar{\tau}(p)$ and b for $\bar{\tau}(p) < t \leq \tau_2 + \varepsilon$, and let $x(t, p)$ and $\lambda(t, p)$ be the corresponding concatenated trajectories and multipliers. This family then forms a C^1-parameterized family $\mathcal{E}$ of broken extremal lifts as defined in [21].

However, by itself this construction does not guarantee that the trajectories do not intersect. Denote by σ_- and σ_+ the flows

$$\sigma_\pm : [\tau_2 - \delta, \tau_2 + \varepsilon] \times P \to [\tau_2 - \delta, \tau_2 + \varepsilon] \times \mathbb{R}^n,\ (t, p) \mapsto (t, x_\pm(t, p))$$

corresponding to the constant controls $u_+ \equiv b$ and $u_- \equiv a$, respectively, integrated from $\mathcal{S}$ with initial, respectively terminal condition $x_+(\tau(p), p) = x_-(\tau(p), p)$ on $\mathcal{S}$. We define each flow over the full interval $[\tau_2 - \delta, \tau_2 + \varepsilon]$. Thus the flow σ_+ extends the flow of x_+ to times $t < \tau(p)$ while the flow σ_- extends the flow of x_- until after the switching.

Definition 4.1 *We say the family Γ_p, $\Gamma_p = (x(\cdot, p), u(\cdot, p), \lambda(\cdot, p))$, $p \in P$, has a* regular transversal crossing *at $\mathcal{S}$ if for all $p \in P$ the derivative of the switching function does not vanish at $\bar{\tau}(p)$, if the flows σ_+ and σ_- have a non-singular Jacobian at $\bar{\tau}(p)$ (i.e. the matrices $\frac{\partial x_\pm}{\partial p}(\bar{\tau}(p), p)$ are nonsingular) and transversally cross $\mathcal{S}$ in the same direction. If instead the flows σ_+ and σ_- cross $\mathcal{S}$ transversally in opposite directions, we say the family Γ_p, $p \in P$, has a* regular transversal fold *at $\mathcal{S}$.*

For an optimal control problem without state space constraints, it is shown in [21] that if the family Γ_p, $p \in P$, has a regular and transversal crossing at $\mathcal{S}$, then it defines a local embedding of the reference trajectory $\Gamma_* = \Gamma_{p_*}$ into a field of extremals. On the other hand, trajectories overlap near the switching surface in case of a regular transversal fold. In this case indeed the switching surface consists of conjugate points and it follows that the restriction of the reference trajectory to an interval $[\tau_2', T]$ with $\tau_2' < \tau_2$ is not optimal. (In [14, Thm. 5.3] these results are developed for a specific problem, but they hold in general.) Thus, and ignoring more degenerate scenarios, here we also assume:

(E) The family Γ_p, $p \in P$, has a regular and transversal crossing at the switching surface $\mathcal{S}$.

If Γ_α is the last boundary arc of the reference trajectory, then computable conditions for (E) to hold are given in [21] and this geometric condition is easily checkable. Essentially, the required transversality conditions for the absence of conjugate points at bang-bang switchings are checked algorithmically involving the computation of a certain Riccati/Lyapunov differential equation.

4.3 Construction of a C^1-parameterized family $\mathcal{E}$ of extremals Γ_p near the exit-junction with state space constraint

So far we have ignored the constraint $M_\alpha = \{(t, x) : h_\alpha(t, x) = 0\}$ that becomes active along the reference trajectory at τ_2. By Proposition 3.2 the graph of the reference trajectory corresponding to the control $u_+ \equiv b$ is transversal to the constraint M_α at $(\tau_2, x_*(\tau_2))$. It thus follows from the implicit function theorem that the equation $h_\alpha(t, x_+(t, p)) = 0$ also has a unique differentiable solution $\tau_2 = \tau_2(p)$ near (τ_2, p_*). Without loss of generality assume it is defined on all of P and let

$$P_+ = \{p \in P : \bar{\tau}(p) > \tau_2(p)\},\quad P_- = \{p \in P : \bar{\tau}(p) < \tau_2(p)\},$$

and

$$P_0 = \{p \in P : \bar{\tau}(p) = \tau_2(p)\}.$$

Geometrically, assumption (E) implies that the dynamics corresponding to the boundary control u_α that keeps the state constraint M_α invariant points to the same side of $\mathcal{S}$ as $u_- = a$ and

$u_+ = b$ (since it is a convex combination of u_- and u_+) and thus the corresponding flow on M_α is transversal to $\mathcal{S}$. It follows that $\mathcal{S}$ and M_α intersect in an $(n-1)$-dimensional embedded submanifold Θ. This submanifold is the image of P_0 under the map

$$\theta : P_0 \to \Theta,\ \tilde{p} \mapsto (\tau_2(\tilde{p}), x_+(\tau_2(\tilde{p}), \tilde{p})).$$

The flow σ_+ along the constant control $u = b$ is a diffeomorpisms and $\theta = \sigma_+ \circ \chi$ where $\chi : P_0 \to \mathbb{R} \times P_0$ is given by $\chi(\tilde{p}) = (\tau_2(\tilde{p}), \tilde{p})$. Thus Θ is the diffeomorphic image of P_0 under θ and P_0 is an $(n-1)$-dimensional embedded submanifold of P that divides P into the two connected components P_+ and P_-. (Recall that we always assume P is chosen sufficiently small.)

When integrating the control $u_+ = b$ backward from points $p \in P_-$, the trajectories $x(\cdot, p)$ will hit the state constraint M_α before they encounter the switching surface while trajectories $x(\cdot, p)$ for $p \in P_+$ will hit the switching surface $\mathcal{S}$ where the control switches to $u_- = a$ thus avoiding M_α entirely (see Fig. 1). In a local synthesis around the reference trajectory we therefore define trajectories $x(\cdot, p)$ for $p \in P_-$ only over the intervals $[\tau_2(p), \tau_2 + \varepsilon]$. These trajectories will not be propagated further backward over the full interval $[t_0, T]$. (In fact, these trajectories are not end pieces of optimal trajectories for the problem (Q) with initial conditions given at time $t = t_0$, at least not in a local synthesis as it is constructed here. Trajectories in an optimal synthesis come off the constraint M_α before they reach these points on M_α. However, if the initial data would be kept varying, as it is done in the construction of a synthesis, then these are optimal trajectories for initial data $(\tau_2(p), x_+(\tau_2(p), p))$ with $p \in P_-$. Thus they give the required embedding of the interior arc of the reference trajectory over $[\tau_2, T]$ needed for the sufficiency argument.) For $p \in P_+$ we continue trajectories past the switching surface $\mathcal{S}$ for some sufficiently small time δ using the control $u_- \equiv a$. We also do this for points on P_0 creating trajectories which only have contact points with the constraint M_α. Since the derivative of the switching function for the reference trajectory is negative at τ_2, this defines extremals if δ is chosen small enough. Then again the trajectories in the field are terminated, i.e. they are only defined over the intervals $[\bar{\tau}(p) - \delta, \tau_2 + \varepsilon]$.

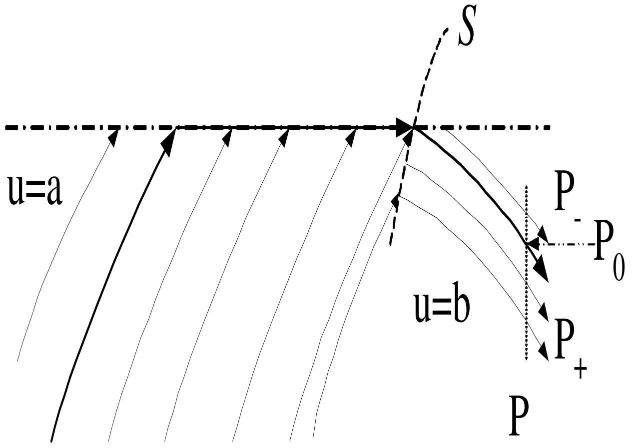

Figure 1: Local synthesis around a boundary arc

4.4 Field of extremals on and off the constraint M_α

The trajectories on M_α are given by the flow corresponding to the continuously differentiable feedback control

$$u_\alpha(t,x) = -\frac{\mathcal{L}_F h_\alpha(t,x)}{\mathcal{L}_G h_\alpha(t,x)}$$

that leaves M_α invariant. Recall that u_α agrees with the reference control u_* along Γ_α, $u_*(t) = u_\alpha(t, x_*(t))$. Therefore, by the continuous dependence of solutions of an ODE on initial conditions, for some small $\delta > 0$ the solution $x_\alpha(t,\tilde{p})$ to the differential equation

$$\dot{x} = f(t,x) + u_\alpha(t,x)g(t,x) \tag{35}$$

with terminal conditions

$$x_\alpha(\tau_2(\tilde{p}),\tilde{p}) = x_+(\tau_2(\tilde{p}),\tilde{p}) \quad \text{for } \tilde{p} \in P_0, \tag{36}$$

will exist over the interval $[\tau_1 - \delta, \tau_2 + \delta]$. Furthermore, since u_α takes values in the interior of the control set over $[\tau_1, \tau_2]$, by keeping δ and P_0 small enough the controls will still take values in the interior of the control set and thus are admissible. Hence this defines a local embedding of the boundary arc Γ_α on the constraint manifold M_α into a field of input-trajectory pairs. However, we do not continue trajectories beyond $\Theta = M_\alpha \cap \mathcal{S}$, i.e. the intersection of the constraint with the switching surface. Thus this flow is only considered for parameters $\tilde{p} \in P_0$ and times $\tau_1 - \delta \leq t \leq \tau_2(\tilde{p})$.

We need to show that this field actually consists of extremals. To this end we integrate the adjoint equation (22) backward from Θ to generate the multiplier $\lambda = \lambda(t,\tilde{p})$, $\tilde{p} \in P_0$, and then define $\nu_\alpha(t,\tilde{p})$ by

$$\nu_\alpha(t,\tilde{p}) = \frac{\lambda(t,\tilde{p})\left(\frac{\partial g}{\partial t}(t, x_\alpha(t,\tilde{p})) + [f,g](t, x_\alpha(t,\tilde{p}))\right)}{\mathcal{L}_G h_\alpha(t, x_\alpha(t,\tilde{p}))}.$$

This guarantees that the switching function remains identically zero while trajectories lie on M_α. It again follows from the continuous dependence on initial data and parameters of a solution to ODE's that $\nu_\alpha(t,\tilde{p})$ is continuous on $\{(t,\tilde{p}) : \tau_1 - \delta \leq t \leq \tau_2(\tilde{p}), \tilde{p} \in P_0\}$ and $\nu_\alpha(t,p_*) = \nu_*(t) > 0$ for $\tau_1 \leq t \leq \tau_2$. By making Θ (or equivalently P) smaller, if necessary, we may therefore assume that there exist positive numbers δ and ϖ such that for all $\tilde{p} \in P_0$ we have $\tau_1 - \delta < \tau_2(\tilde{p}) < \tau_2 + \delta$ and

$$\nu_\alpha(t,\tilde{p}) \geq \varpi \text{ on } \hat{D}_\alpha = \{(t,\tilde{p}) : \tau_1 - \delta \leq t \leq \tau_2 + \delta,\ \tilde{p} \in P_0\}. \tag{37}$$

In particular, on M_α we thus have a field $\mathcal{F}_\partial$ of boundary-arc extremals for the problem (Q) which contains Γ_α.

Condition (37) also guarantees that for some short duration we obtain extremals when we integrate the control $u_- \equiv a$ backward from any point on M_α covered by the field $\mathcal{F}_\partial$. For, since u_α takes values in the interior of the control set, on the constraint the derivative of the switching function is identically zero and given by (21),

$$\dot{\Phi}(t,\tilde{p}) = \lambda(t,\tilde{p})\left(\frac{\partial g}{\partial t}(t, x_\alpha(t,\tilde{p})) + [f,g](t, x_\alpha(t,\tilde{p}))\right) - \nu_\alpha(t,\tilde{p})\mathcal{L}_G h_\alpha(t, x_\alpha(t,\tilde{p})).$$

Furthermore, with our normalization $\mathcal{L}_G h_\alpha$ is negative. Thus, if we integrate the control $u_- \equiv a$ backward from a point $(\tilde{\tau}_1, \tilde{x}) = (\tilde{\tau}_1, x_\alpha(\tilde{\tau}_1, \tilde{p}))$ on M_α, $(\tilde{\tau}_1, \tilde{p}) \in \hat{D}_\alpha$, then analogously to (27)

$$\dot{\Phi}(\tilde{\tau}_1-, x_\alpha(\tilde{\tau}_1, \tilde{p})) = \lambda(\tilde{\tau}_1, \tilde{p}) \left(\frac{\partial g}{\partial t}(\tilde{\tau}_1, x_\alpha(\tilde{\tau}_1, \tilde{p})) + [f, g](\tilde{\tau}_1, x_\alpha(\tilde{\tau}_1, \tilde{p})) \right) < 0.$$

Thus the control $u_- = a$ satisfies the minimality condition (17). Since all the bounds are uniform over D_α, the corresponding trajectories will remain extremal for some positive duration $\delta > 0$ independent of the initial condition. Thus the positivity of the multiplier ν_α guarantees that we can integrate the system backward from points on the constraint manifold M_α using the control $u_- = a$ and get extremals. Finally, the flow σ_- consists of solutions to the differential equation

$$\dot{x} = f(t, x) + ag(t, x), \quad \dot{t} = 1,$$

with terminal conditions $q = (\tilde{\tau}_1, x(\tilde{\tau}_1, \tilde{p}))$, $\tilde{p} \in P_0$, on the manifold M_α. By Proposition 3.2 this flow is everywhere transversal to M_α and it follows from the uniqueness of solutions that the corresponding trajectories cannot intersect. Hence this construction gives the required local embedding of Γ_α.

4.5 Parametrization of the field

For later reference we summarize the parametrization of the field $\mathcal{F}$ around the reference trajectory $\Gamma = (x_*, u_*)$ constructed above. In fact, we will define a stratification of the region R covered by the extremals of the field and give a diffeomorphic parametrization for each stratum. Recall that a C^r-*stratification* $\mathcal{A} = \{A_i : i \in I\}$ of a C^r-manifold M is a locally finite family of connected C^r-embedded submanifolds A_i of M, $i \in I$, which satisfies the frontier axiom. An element of $\mathcal{A}$ is called a stratum. (A set A is the locally finite union of the A_i, $i \in I$, if $A = \cup_{i \in I} A_i$ and every compact subset K of M only intersects finitely many of the A_i. For a subset A of M its frontier, $Fron A$, is the set of all boundary-points of A in M which do not lie in A. The collection $\mathcal{A}$ is said to satisfy the frontier axiom if whenever A_i and A_j are elements of $\mathcal{A}$, $A_i \neq A_j$, such that $A_i \cap Clos A_j \neq \emptyset$, then $A_i \subset Fron A_j$ and $\dim A_i < \dim A_j$, i.e. for every $A \in \mathcal{A}$ the frontier of A is a union of members of $\mathcal{A}$ of smaller dimensions.) We parameterize each portion of the field over some domain $D_{(\cdot)}$ in (t, p)-space with $u(t, p)$ and $x(t, p)$ denoting the corresponding controls and trajectories.

$\boxed{\mathcal{T}}$: The terminal points for the construction are given by the submanifold

$$N_0 = \{(t, x) : t = \tau_2 + \varepsilon, \ x \in P\}.$$

$\boxed{\mathcal{F}_1}$: Graphs of trajectories corresponding to the control $u = b$: set

$$D_1 = \{(t, p) : p \in P, \ \max(\tau_2(p), \bar{\tau}(p)) < t < \tau_2 + \varepsilon\}$$

and on D_1 define the control by $u(t, p) \equiv b$ and $x(t, p)$ as the corresponding trajectory with terminal value $x(\tau_2 + \varepsilon, p) = p$. For $p \in P_-$ the maximum is attained for $\tau_2(p)$ and those trajectories will not be propagated further backward in time. For $p \in P_+$ the maximum is attained for $\bar{\tau}(p)$ and there the controls switch to $u = a$. Set $R_1 = \sigma(D_1)$, the image under of flow defined by the graphs of the trajectories. Strata in the frontier of R_1 are given by N_0, the terminal points, and the following lower dimensional strata:

$\boxed{\mathcal{G}_1}$: Points on the constraint M_α that form initial points for the trajectories of $\mathcal{F}_1$ that end in P_-,

$$N_1 = \{(t,x) = (\tau_2(p), x(\tau_2(p),p)),\ p \in P_-\}.$$

These trajectories will not be propagated backward in time, but provide the required embedding for the exit arc.

$\boxed{\mathcal{G}_2}$: Points on the constraint M_α where the trajectories in the field have only contact points with M_α,

$$N_2 = \{(t,x) = (\tau_2(\tilde{p}), x(\tau_2(\tilde{p}),\tilde{p})),\ \tilde{p} \in P_0\}.$$

Recall that $\tau_2(\tilde{p}) = \bar{\tau}(\tilde{p})$ for $\tilde{p} \in P_0$ and hence N_2 is a codimension 2 submanifold that also lies in the frontier of N_1.

$\boxed{\mathcal{G}_3}$: Points on the switching surface,

$$\mathcal{S} = \{(t,x) = (\bar{\tau}(p), x(\bar{\tau}(p),p)),\ p \in P_+\}.$$

$\boxed{\mathcal{F}_2}$: Trajectories in $\mathcal{F}_1$ that end in P_+ do not meet the constraint M_α and the controls switch to $u = b$ as the trajectories cross $\mathcal{S}$ transversally. This patch extends the parametrization of these trajectories. By shrinking P, if necessary, we may assume that there exists a $\delta > 0$ so that $\tau_2 - \delta < \bar{\tau}(p)$ for all $p \in P_+$ and then define

$$D_2 = \{(t,p) : p \in P_+,\ \tau_2 - \delta < t < \bar{\tau}(p)\}.$$

On D_2 the control is given by $u(t,p) = a$ and $x(t,p)$ is the corresponding trajectory with terminal value given by $x(\bar{\tau}(p),p)$ as constructed for $\mathcal{S}$. Let $R_2 = \sigma(D_2)$, the image under of flow defined by the graphs of the trajectories. Because of assumption (E), the two bang flows cross the switching surface transversally, it follows that R_1 and R_2 are disjoint if δ is chosen small enough.

$\boxed{\mathcal{G}_4}$: Graphs of trajectories that only have a contact point with M_α at points in N_2: let

$$D_c = \{(t,\tilde{p}) : \tilde{p} \in P_0,\ \tau_2 - \delta < t < \tau_2(\tilde{p})\}$$

with control and trajectories defined as for $\mathcal{F}_2$ and denote the image under σ by $\mathcal{C}$, $\mathcal{C} = \sigma(D_c)$.

$\boxed{\mathcal{M}}$: Graphs of trajectories on the constraint M_α away from $\mathcal{S}$: we parameterize these trajectories over some set

$$D_m = \{(t,\tilde{p}) : \tilde{p} \in P_0,\ \tau_1 - \vartheta < t < \tau_2(\tilde{p})\}$$

where we select a suitable small positive number ϑ so that the boundary trajectories are defined and are extremal over the interval $[\tau_1 - \vartheta, \tau_2(\tilde{p})]$. For this patch the trajectories $x(t,p)$ are the solutions to

$$\dot{x} = f(t,x) - \frac{\mathcal{L}_F h_\alpha(t,x)}{\mathcal{L}_G h_\alpha(t,x)} g(t,x)$$

with terminal conditions given by $x(\tau_2(\tilde{p}),\tilde{p})$ at time $\tau_2(\tilde{p})$ with $\tilde{p} \in P_0$. The control $u(t,\tilde{p})$ is the corresponding open-loop control

$$u(t,\tilde{p}) = -\frac{\mathcal{L}_F h_\alpha(t, x(t,\tilde{p}))}{\mathcal{L}_G h_\alpha(t, x(t,\tilde{p}))}.$$

The image $R_m = \sigma(D_m)$ is the part of M_α covered by these trajectories.

$\boxed{\mathcal{F}_3}$: The last portion of the synthesis is obtained by integrating the control $u = a$ backward from points on $R_m \subset M_\alpha$. We parameterize these trajectories through the values $\tilde{p}$ in the $(n-1)$-dimensional manifold P_0 that define the trajectories on the constraint and an entry time $\tilde{\tau}_1$ onto this trajectory. Because of the qualitative difference in the type of parametrization we now call the parameter q, i.e. $q = (\tilde{\tau}_1, \tilde{p})$ with $\tilde{p} \in P_0$ and $\tau_1 - \vartheta \leq \tilde{\tau}_1 < \tau_2(\tilde{p})$. As domain we take

$$D_3 = \{(t,q) : q = (\tilde{\tau}_1, \tilde{p}),\ \tilde{\tau}_1 - \delta \leq t < \tilde{\tau}_1\}$$

where δ is small enough that the trajectory $x(t,q)$ corresponding to control $u(t,q) = a$ with terminal value $x(\tilde{\tau}_1, \tilde{p})$ at time $\tilde{\tau}_1$ is an extremal. Again we set $R_3 = \sigma(D_3)$.

For each piece in the stratification the given parametrization of the trajectories,

$$\sigma : D \to [\tau_1 - \vartheta, \tau_2 + \varepsilon] \times \mathbb{R}^n, \quad (t,p) \mapsto (t, x(t,p)),$$

is a diffeomorphism and since the images do not overlap, this parametrization also is injective overall. Note that the constructed trajectories are unique forward in time (but not backward in time since the trajectories from $\mathcal{F}_3$ collapse onto the codimension 1 manifold M_α as they enter the stratum $\mathcal{M}$) and they define a memoryless synthesis of extremals with corresponding multipliers given by $\lambda(t,p)$ and $\nu(t,p)$. Thus this defines a local field $\mathcal{F}$ of extremals. The strata in (t,x)-space are illustrated in Fig. 2.

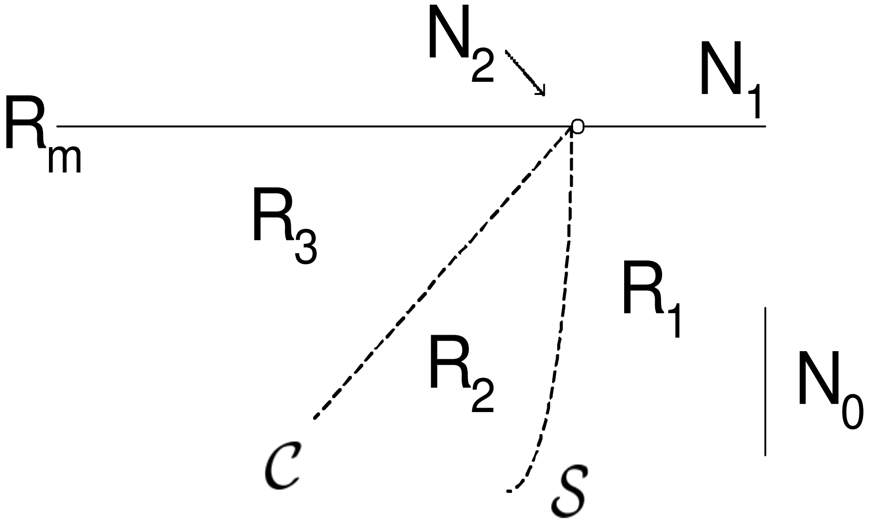

Figure 2: Stratification of the local field $\mathcal{F}$

5 The Value-function of the Field

Our aim is to prove that the existence of the field constructed in section 4 implies the strong local optimality of the reference trajectory. For problems without state space constraints these results are classical, but results of this type have not been established for problems with state space constraints. We first need to establish regularity properties of the value function of the field $\mathcal{F}$ which then allow to make this argument. We again restrict the analysis to the local field around the boundary arc. In view of assumption (D) it suffices to consider the following optimal control problem:

$(\hat{\mathbf{Q}})$ minimize $J(u) = V^*(\tau_2+\varepsilon, x(\tau_2+\varepsilon))$ over all Lebesgue measurable functions u, $u : [t, \tau_2 + \varepsilon] \to [a, b]$, with values in the compact interval $[a, b] \subset \mathbb{R}$ subject to the dynamics $\dot{x}(s) = f(s, x) + ug(s, x)$ with a given value $x(t)$ at time t, and state space constraint $h_\alpha(s, x) \le 0$.

The term $V^*(\tau_2 + \varepsilon, x(\tau_2 + \varepsilon))$ represents the value or cost of the optimal control problem (Q) over the final portion $[\tau_2 + \varepsilon, T]$ with initial data $(\tau_2 + \varepsilon, x(\tau_2 + \varepsilon))$. This then allows to obtain optimality over the full interval $[t, T]$ by combining the local field $\mathcal{F}$ with a synthesis over $[\tau_2 + \varepsilon, T]$.

For the field $\mathcal{F}$ define the associated value function $V = V^{\mathcal{F}}$ at $(t, x) \in R$ as the value of the cost for the trajectory which starts at x at time t. Since the local synthesis in the field is memoryless, there exists a unique control-trajectory pair defined over the interval $[t, \tau_2 + \varepsilon]$ that starts at (t, x). For a problem formulation in Meyer form the cost only depends on the parameter value p respectively $\tilde{p}$ characterizing the actual trajectory through (t, x), but not on the time t.

We need to establish continuity and differentiability properties of the function $V^{\mathcal{F}}$ in (t, x)-space. Continuity immediatly follows from the structure of the field.

Proposition 5.1 *The value function $V^{\mathcal{F}}(t, x)$ is continuous.*

Proof. Given (t, x), let $\{(t_n, x_n)\}_{n\in\mathbb{N}}$ be a sequence so that $(t_n, x_n) \to (t, x)$. Denote the corresponding input-trajectory pairs of the field $\mathcal{F}$ that start at these points by (ξ, η) and (ξ_n, η_n), respectively. Without loss of generality assume all functions are defined on a common interval $[t - \delta, t_2 + \varepsilon]$ for some small $\delta > 0$. It follows from the structure of the field $\mathcal{F}$ that the trajectories ξ_n converge to ξ uniformly on $[t - \delta, t_2 + \varepsilon]$. Thus, since $V^*(\tau_2 + \varepsilon, \cdot)$ is continuous, we get

$$V^{\mathcal{F}}(t_n, x_n) = V^*(\tau_2 + \varepsilon, \xi_n(\tau_2 + \varepsilon)) \to V^*(\tau_2 + \varepsilon, \xi(\tau_2 + \varepsilon)) = V^{\mathcal{F}}(t, x).$$

□

Theorem 5.1 *The value function $V^{\mathcal{F}}$ is continuously differentiable on each of the open strata R_1, R_2, and R_3. Given $(t, x) \in R_i$ with corresponding parametrization $x = x(t, p)$ as defined in section 4.5, we have*

$$\frac{\partial V}{\partial x}(t, x) = \lambda(t, p) \tag{38}$$

and

$$\frac{\partial V}{\partial t}(t, x) = -H(t, \lambda(t, p), x(t, p), u(t, p)). \tag{39}$$

Proof. This statement is classical for the regions R_1 and R_2 (e.g. [2]) and is only included for further reference: The flow corresponding to the constant control $u \equiv b$, $\sigma : (t, p) \to (t, x(t, p))$, is a C^1-diffeomorphisms on the set D_1 and the function

$$p \mapsto C(p) = V^*(\tau_2 + \varepsilon, p)$$

is continuously differentiable by assumption. Hence $V^{\mathcal{F}} = C \circ \sigma^{-1}$ is continuously differentiable on R_1. Trajectories emanating from P_+ have a bang-bang switch as they transversally cross the switching surface $\mathcal{S}$ and avoid M_α entirely. These trajectories do not meet the constraint and thus the results from [21] for optimal control problems without state space constraints apply. Hence, by [21, Theorem 2.12] $V^{\mathcal{F}}$ is continuously differentiable on the image R_2. Equations (38) and (39) then also follow for the flow of broken extremals from the Shadow Price Lemma in [21]. The essential identity in the proof of these relations is

$$\frac{\partial C}{\partial p}(p) = \lambda(t,p)\frac{\partial x}{\partial p}(t,p) \tag{40}$$

which allows to identify $\frac{\partial V}{\partial x}(t, x(t,p))$ with $\lambda(t,p)$. For points $(t,p) \in D_1 \cup D_2$ this directly follows from the results in [21]: the flows corresponding to the constant controls are diffeomorphisms and by assumption (E) they cross the switching surface transversally. Thus the Shadow-Price lemma [21, Lem. 2.4] remains valid showing (40).

The novel feature in the construction is to show that these relations remain valid also for trajectories that have boundary arcs. Given $(t,x) \in R_3$, there exists a unique parameter q, $q = (\tilde{\tau}_1, \tilde{p})$, $\tilde{p} \in P_0$, $\tilde{\tau}_1 < \tau_2(\tilde{p})$, so that $x = x(t,q)$. The corresponding trajectory $x(s;q)$, $t \le s \le \tau_2 + \varepsilon$, is given by

$$x(s;q) = \begin{cases} x_+(s,\tilde{p}) & \text{for } \tau_2(\tilde{p}) \le s \le \tau_2 + \varepsilon \\ x_\alpha(s,\tilde{p}) & \text{for } \tilde{\tau}_1 \le s \le \tau_2(\tilde{p}) \\ x_-(s,(\tilde{\tau}_1,\tilde{p})) & \text{for } t \le s \le \tilde{\tau}_1 \end{cases} . \tag{41}$$

Here, as before, $x_+(s,\tilde{p})$ denotes the trajectory on the interval $\tau_2(\tilde{p}) \le s \le \tau_2 + \varepsilon$ corresponding to the control $u = b$ with terminal condition $x_+(\tau_2 + \varepsilon, \tilde{p}) = \tilde{p}$. Along the boundary segment, for $s \in (\tilde{\tau}_1, \tau_2(\tilde{p}))$, $x(s;q)$ is given by the trajectory $x_\alpha(s,\tilde{p})$ corresponding to the smooth feedback $u_\alpha(s,x)$ defined by (9) with value $x_+(\tau_2(\tilde{p}),\tilde{p})$ at $\tau_2(\tilde{p})$ and on the final interval $x(s;q)$ is given by the trajectory corresponding to control $u = a$ with terminal value $x_\alpha(\tilde{\tau}_1,\tilde{p})$ at time $\tilde{\tau}_1$. We also denote by $\lambda(s;q)$ the corresponding adjoint variable. Since entry and exit to the constraint are transversal, the adjoint $\lambda(s;q)$ is absolutely continuous on $[t, \tau_2 + \varepsilon]$. Note that for times $s \ge \tilde{\tau}_1$ both the trajectories and the adjoint variables in this parametrization only depend on the parameter $\tilde{p} \in P_0$, but not on $\tilde{\tau}_1$. In particular, we have at the final time

$$x(\tau_2 + \varepsilon; q) = x_+(\tau_2 + \varepsilon; \tilde{p}) = \tilde{p} \tag{42}$$

and by construction

$$\lambda(\tau_2 + \varepsilon; q) = \lambda(\tau_2 + \varepsilon; (\tilde{\tau}_1, \tilde{p})) = \frac{\partial V^*}{\partial x}(\tau_2 + \varepsilon, \tilde{p}) = \frac{\partial V^*}{\partial x}(\tau_2 + \varepsilon, x(\tau_2 + \varepsilon; q)). \tag{43}$$

The parametrized cost only depends on the parameter value $\tilde{p} \in P_0$, but not $\tilde{\tau}_1$, and we have

$$C(q) = C(\tilde{\tau}_1, \tilde{p}) = V^*(\tau_2 + \varepsilon, \tilde{p}) = V^*(\tau_2 + \varepsilon, x(\tau_2 + \varepsilon; q)). \tag{44}$$

Thus we have

$$\begin{aligned} \frac{\partial C}{\partial q}(q) &= \left(\frac{\partial C}{\partial \tilde{\tau}_1}(\tilde{\tau}_1, \tilde{p}), \frac{\partial C}{\partial \tilde{p}}(\tilde{\tau}_1, \tilde{p})\right) = \left(0, \frac{\partial V^*}{\partial x}(\tau_2 + \varepsilon, \tilde{p})\frac{\partial x}{\partial \tilde{p}}(\tau_2 + \varepsilon; q)\right) \\ &= \left(0, \lambda(\tau_2 + \varepsilon; q)\frac{\partial x}{\partial \tilde{p}}(\tau_2 + \varepsilon; q)\right). \end{aligned} \tag{45}$$

Lemma 5.1 *The function*

$$s \mapsto \lambda(s;q)\frac{\partial x}{\partial \tilde{p}}(s;q) \tag{46}$$

is constant over the interval $[t, \tau_2 + \varepsilon]$.

Proof of the Lemma. We first show that the time derivative of (46) vanishes on each of the open intervals $(t, \tilde{\tau}_1)$, $(\tilde{\tau}_1, \tau_2(\tilde{p}))$ and $(\tau_2(\tilde{p}), \tau_2 + \varepsilon)$. Along the boundary interval, i.e. for $s \in (\tilde{\tau}_1, \tau_2(\tilde{p}))$, we have that

$$\begin{aligned}
\frac{d}{ds}\left(\lambda(s;q)\frac{\partial x_\alpha}{\partial \tilde{p}}(s,\tilde{p})\right) &= \dot{\lambda}(s;q)\frac{\partial x_\alpha}{\partial \tilde{p}}(s,\tilde{p}) + \lambda(s;q)\frac{\partial^2 x_\alpha}{\partial t \partial \tilde{p}}(s,\tilde{p}) \\
&= \left(-\lambda\left(f_x + u_\alpha g_x\right) - \nu\frac{\partial h_\alpha}{\partial x}\right)\frac{\partial x_\alpha}{\partial \tilde{p}} + \lambda\left(\left(f_x + u_\alpha g_x\right) + g\frac{\partial u_\alpha}{\partial x}\right)\frac{\partial x_\alpha}{\partial \tilde{p}} \\
&= \left(-\nu\frac{\partial h_\alpha}{\partial x} + \lambda g\frac{\partial u_\alpha}{\partial x}\right)\frac{\partial x_\alpha}{\partial \tilde{p}} = -\nu\frac{\partial h_\alpha}{\partial x}\frac{\partial x_\alpha}{\partial \tilde{p}} + H_u\frac{\partial u_\alpha}{\partial x}\frac{\partial x_\alpha}{\partial \tilde{p}}
\end{aligned}$$

where H denotes the Hamiltonian (11). As in [21], it follows from the minimum condition (12) that

$$H_u\frac{\partial u_\alpha}{\partial x}\frac{\partial x_\alpha}{\partial \tilde{p}} \equiv 0.$$

(The function

$$h(\omega) = \lambda(s;q)\left(f(s, x_\alpha(s,\tilde{p})) + u_\alpha(s, x_\alpha(s,\omega)g(s, x_\alpha(s,\tilde{p}))\right)$$

is differentiable at $\omega \in P_0$ and has a local minimum at $\omega = \tilde{p}$.) Furthermore, since the trajectories lie on the constraint M_α, we have $h_\alpha(s, x_\alpha(s,\tilde{p})) \equiv 0$ and thus also

$$\frac{\partial h_\alpha}{\partial x}\frac{\partial x_\alpha}{\partial \tilde{p}} \equiv 0.$$

Hence

$$\frac{d}{ds}\left(\lambda(s;q)\frac{\partial x_\alpha}{\partial \tilde{p}}(s,\tilde{p})\right) = 0.$$

This relation follows analogously for the time intervals $[t, \tilde{\tau}_1)$ and $(\tau_2(\tilde{p}), \tau_2 + \varepsilon]$ when the constraint is not active by setting $\nu = 0$ (see also [21, Lem. 2.5]).

This computation shows that the expression (46) is constant over the open intervals $(\tau_1 - \varepsilon, \tilde{\tau}_1)$, $(\tilde{\tau}_1, \tau_2(\tilde{p}))$ and $(\tau_2(\tilde{p}), \tau_2 + \varepsilon)$. But the partial derivatives with respect to the parameter $\tilde{p}$ are discontinuous at the exit time $\tau_2(\tilde{p})$. However, these jumps in the partial derivatives match up (see also [21, Lem. 2.5]). As before, denote by x_α and x_+ the trajectories to the left and to the right of $\tau_2(\tilde{p})$. Recall that we have $x_\alpha(\tau_2(\tilde{p}), \tilde{p}) = x_+(\tau_2(\tilde{p}), \tilde{p})$ for all $\tilde{p} \in P_0$. It follows from a simple calculus argument [21, Lem. 2.6] that there exists a continuous function κ defined on P_0 so that

$$grad\ x_\alpha(\tau_2(\tilde{p}), \tilde{p}) = grad\ x_+(\tau_2(\tilde{p}), \tilde{p}) + \kappa(\tilde{p})(1, -\frac{\partial \tau_2}{\partial \tilde{p}}(\tilde{p})),$$

where *grad* denotes the gradient in the variables $(t, \tilde{p})$. With $u_\alpha^2 = u_\alpha(\tau_2(\tilde{p}), x_\alpha(\tau_2(\tilde{p}), \tilde{p}))$ it follows from the t-derivatives that

$$\kappa(\tilde{p}) = (u_\alpha^2 - b)g(\tau_2(\tilde{p}), x_\alpha(\tau_2(\tilde{p}), \tilde{p}))$$

and hence

$$\frac{\partial x_\alpha}{\partial \tilde{p}}(\tau_2(\tilde{p}),\tilde{p}) - \frac{\partial x_+}{\partial \tilde{p}}(\tau_2(\tilde{p}),\tilde{p}) = (b-u_\alpha^2)g(\tau_2(\tilde{p}),x_\alpha(\tau_2(\tilde{p}),\tilde{p}))\frac{\partial \tau_2}{\partial \tilde{p}}(\tilde{p}).$$

The multiplier $\lambda = \lambda(\cdot, q)$ remains continuous at the junction and so does the switching function $\Phi = \lambda g$. Since $\Phi(\tau_2(\tilde{p}),\tilde{p}) = 0$, we therefore get the desired relation at the exit-junction:

$$\lambda(\tau_2, q)\left(\frac{\partial x_\alpha}{\partial \tilde{p}}(\tau_2(\tilde{p}),\tilde{p}) - \frac{\partial x_+}{\partial \tilde{p}}(\tau_2(\tilde{p}),\tilde{p})\right) = (b-u_\alpha^2)\lambda(\tau_2(\tilde{p}),q)g(\tau_2(\tilde{p}),x_\alpha(\tau_2(\tilde{p}),\tilde{p}))\frac{\partial \tau_2}{\partial \tilde{p}}(\tilde{p}) = 0.$$

At the entry-junction the partial derivatives remain continuous: for $t \le s \le \tilde{\tau}_1$, the trajectory $x(s;q)$ is the solution $x_-(s,(\tilde{\tau}_1,\tilde{p}))$ to the differential equation $\dot{x} = f(t,x) + ag(t,x)$ with terminal condition given by $x_\alpha(\tilde{\tau}_1,\tilde{p})$ at time $\tilde{\tau}_1$. Differentiating the identity $x_-(\tilde{\tau}_1;(\tilde{\tau}_1,\tilde{p})) = x_\alpha(\tilde{\tau}_1,\tilde{p})$ that defines the terminal condition for the trajectory x_- on the constraint with respect to $\tilde{p}$ yields

$$\frac{\partial x_-}{\partial \tilde{p}}(\tilde{\tau}_1,(\tilde{\tau}_1,\tilde{p})) = \frac{\partial x_\alpha}{\partial \tilde{p}}(\tilde{\tau}_1,\tilde{p})$$

and thus

$$\lambda(\tilde{\tau}_1,q)\frac{\partial x_-}{\partial \tilde{p}}(\tilde{\tau}_1,(\tilde{\tau}_1,\tilde{p})) = \lambda(\tilde{\tau}_1,q)\frac{\partial x_\alpha}{\partial \tilde{p}}(\tilde{\tau}_1,\tilde{p}).$$

This proves that the function $s \mapsto \lambda(s;q)\frac{\partial x}{\partial \tilde{p}}(s;q)$ is constant over $[t,\tau_2+\varepsilon]$. □

Lemma 5.2

$$\lambda(s;q)\frac{\partial x}{\partial \tilde{\tau}_1}(s;q) \equiv 0 \quad \text{for all} \quad s \in [t,\tau_2+\varepsilon]. \tag{47}$$

Proof of the Lemma. This is clear for the intervals $(\tilde{\tau}_1,\tau_2(\tilde{p}))$, $(\tau_2(\tilde{p}),\tau_2+\varepsilon)$, and the junction at $\tau_2(\tilde{p})$ since neither $x_+(s,\tilde{p})$ nor $x_\alpha(s,\tilde{p})$ depend on $\tilde{\tau}_1$ and thus $\frac{\partial x}{\partial \tilde{\tau}_1}(s;q) \equiv 0$ on these intervals. Hence we only need to consider the interval $[t,\tilde{\tau}_1]$. Differentiating the identity $x_-(\tilde{\tau}_1;(\tilde{\tau}_1,\tilde{p})) = x_\alpha(\tilde{\tau}_1,\tilde{p})$ now with respect to $\tilde{\tau}_1$, we get that

$$\frac{\partial x_-}{\partial t}(\tilde{\tau}_1;(\tilde{\tau}_1,\tilde{p})) + \frac{\partial x_-}{\partial \tilde{\tau}_1}(\tilde{\tau}_1;(\tilde{\tau}_1,\tilde{p})) = \frac{\partial x_\alpha}{\partial t}(\tilde{\tau}_1,\tilde{p}),$$

i.e. in terms of the time derivatives, and setting $u_\alpha^1 = u_\alpha(\tilde{\tau}_1,x_\alpha(\tilde{\tau}_1,\tilde{p}))$ and $q = (\tilde{\tau}_1,\tilde{p})$, we have

$$\begin{aligned}\frac{\partial x_-}{\partial \tilde{\tau}_1}(\tilde{\tau}_1;q) &= \frac{dx_\alpha}{dt}(\tilde{\tau}_1,\tilde{p}) - \frac{dx_-}{dt}(\tilde{\tau}_1;q)\\ &= f(\tilde{\tau}_1,x_\alpha(\tilde{\tau}_1,\tilde{p})) + u_\alpha^1 g(\tilde{\tau}_1,x_\alpha(\tilde{\tau}_1,\tilde{p})) - f(\tilde{\tau}_1,x_-(\tilde{\tau}_1,q)) - ag(\tilde{\tau}_1,x_-(\tilde{\tau}_1,q))\\ &= (u_\alpha^1 - a)g(\tilde{\tau}_1,x_\alpha(\tilde{\tau}_1,p)).\end{aligned}$$

Thus, since the switching function $\Phi = \lambda g$ vanishes at the entry-junction, we also have

$$\lambda(\tilde{\tau}_1;q)\frac{\partial x_-}{\partial \tilde{\tau}_1}(\tilde{\tau}_1;q) = (u_\alpha^1 - a)\lambda(\tilde{\tau}_1;q)g(\tilde{\tau}_1,x_\alpha(\tilde{\tau}_1,p)) = 0.$$

Finally, on the interval $[t,\tilde{\tau}_1)$ this quantity remains constant since the adjoint equation is the adjoint to the variational equation,

$$\begin{aligned}\frac{d}{ds}\left(\lambda(s;q)\frac{\partial x_-}{\partial \tilde{\tau}_1}(s,\tilde{p})\right) &= \dot{\lambda}(s;q)\frac{\partial x_-}{\partial \tilde{\tau}_1}(s,\tilde{p}) + \lambda(s;q)\frac{\partial^2 x_-}{\partial t \partial \tilde{\tau}_1}(s,\tilde{p})\\ &= -\lambda\left(f_x + ag_x\right)\frac{\partial x_-}{\partial \tilde{\tau}_1} + \lambda\left(f_x + ag_x\right)\frac{\partial x_-}{\partial \tilde{\tau}_1} = 0.\end{aligned}$$

This proves the second lemma. □

Summarizing, from these lemmas and (45) we therefore have

$$\begin{aligned}\lambda(t;q)\frac{\partial x}{\partial q}(t;q) &= \lambda(t;q)\left(\frac{\partial x}{\partial \tilde{\tau}_1}(t;(\tilde{\tau}_1,\tilde{p})), \frac{\partial x}{\partial \tilde{p}}(t;(\tilde{\tau}_1,\tilde{p}))\right)\\ &= \left(0, \lambda(\tau_2+\varepsilon;q)\frac{\partial x}{\partial \tilde{p}}(\tau_2+\varepsilon;q)\right) = \frac{\partial C}{\partial q}(q). \qquad (48)\end{aligned}$$

From this the relation (38) easily follows for points $(t,x)\in R_3$: Since the dynamics $f(t,x)+ag(t,x)$ is everywhere transversal to the constraint manifold M_α, the flow σ of trajectories corresponding to the constant control $u\equiv a$ is a diffeomorphism and the value function $V^{\mathcal{F}}$ satisfies $V^{\mathcal{F}}\circ\sigma = C$ where C denotes the parameterized cost defined in (44),

$$V^{\mathcal{F}}(t,x_-(t,q)) = C(q).$$

Hence $V^{\mathcal{F}} = C\circ\sigma^{-1}$ is differentiable on the open set R_3 and we have at $(t,x)=(t,x_-(t,q))\in R_3$

$$\left(\frac{\partial V^{\mathcal{F}}}{\partial t}(t,x), \frac{\partial V^{\mathcal{F}}}{\partial x}(t,x)\right)\begin{pmatrix} 1 & 0\\ \dot{x}_-(t,q) & \frac{\partial x_-}{\partial q}(t,q)\end{pmatrix} = \left(0, \frac{\partial C}{\partial q}(q)\right).$$

Thus, in view of (48) the second equation yields

$$\frac{\partial V^{\mathcal{F}}}{\partial x}(t,x) = \frac{\partial V^{\mathcal{F}}}{\partial x}(t,x_-(t,p)) = \lambda(t,p)$$

and then it follows from the first equation that

$$\frac{\partial V^{\mathcal{F}}}{\partial t}(t,x) = -\lambda(t,p)\left(f(t,x_-(t,p))+ag(t,x_-(t,p))\right) = -H(t,\lambda(t,p),x_-(t,p),a).$$

This proves the proposition for points in R_3. □

In general the value function $V^{\mathcal{F}}$ need not be differentiable on the lower dimensional strata $\mathcal{C}$ and $\mathcal{S}$. While the restrictions to these $(n-1)$-dimensional submanifolds are continuously differentiable, and while the derivatives of $V^{\mathcal{F}}$ from the sets R_1, R_2, and R_3 have well-defined extensions to $\mathcal{C}$ and $\mathcal{S}$, these in general do not match up in the direction transversal to these surfaces. However, as will be shown in Theorem 7.1, the differentiability on these surfaces is not necessary to prove the optimality of the synthesis. We only need the following statement:

Corollary 5.1 *On the open sets R_1, R_2, and R_3 the value function $V^{\mathcal{F}}$ associated with the field $\mathcal{F}$ satisfies the Hamilton-Jacobi-Bellman equation, i.e.*

$$V_t^{\mathcal{F}}(t,x) + \min_{a\le u\le b}\{V_x^{\mathcal{F}}(t,x)f(t,x)+ug(t,x)\} \equiv 0. \qquad (49)$$

Proof. It is well-known that this is a direct consequence of the relations (38) and (39). For, at any $(t,x)\in R_1\cup R_2\cup R_3$ there exists a unique patch $\mathcal{F}_i$ that parameterizes (t,x) and with (t,p) denoting the corresponding parameter, by the minimum condition (12) we have for any control value $v\in[a,b]$

$$\begin{aligned}&V_t^{\mathcal{F}}(t,x) + V_x^{\mathcal{F}}(t,x)\left(f(t,x)+vg(t,x)\right)\\ &= V_t^{\mathcal{F}}(t,x(t,p)) + V_x^{\mathcal{F}}(t,x(t,p))\left(f(t,x(t,p))+vg(t,x(t,p))\right)\\ &= -H(t,\lambda(t,p),x(t,p),u(t,p)) + \lambda(t,p)\left(f(t,x(t,p))+vg(t,x(t,p))\right)\\ &= H(t,\lambda(t,p),x(t,p),v) - H(t,\lambda(t,p),x(t,p),u(t,p)) \ge 0\end{aligned}$$

and equality holds for $v=u(t,p)$. □

6 Stability of the local synthesis

The synthesis $\mathcal{F}$ constructed above has a strong stability property if the value of the constraint is relaxed. Let

$$\Omega = \{(t,x) : \tau_1 - \vartheta \le t \le \tau_2 + \varepsilon,\ ||x - x_*(t)||_\infty < \omega\}$$

and pick ω small enough so that the admissible portion

$$\Omega_{ad} = \{(t,x) \in \Omega : h_\alpha(t,x) \le 0\}$$

is covered by the extremals in $\mathcal{F}$. We show that for small $\Delta > 0$ these extremals can be embedded into a field $\mathcal{F}^\Delta$ of extremals for the problem (Q_Δ) where the state space constraints have been relaxed to

$$\Omega_{ad}^\Delta = \{(t,x) \in \Omega : h_\alpha(t,x) \le \Delta\}.$$

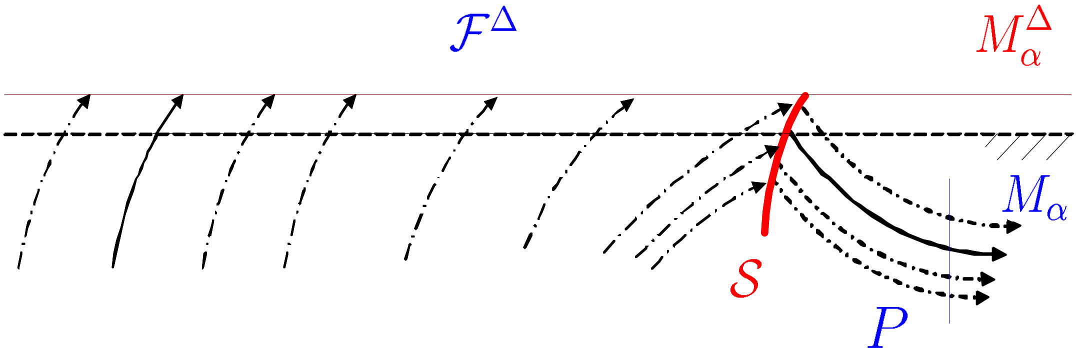

Figure 3: Stability of the local synthesis

Proposition 6.1 *Assume conditions* (A) - (E) *hold. Then, for* $\Delta > 0$ *sufficiently small, there exists a local field* $\mathcal{F}^\Delta$ *of extremals for the problem* (Q_Δ) *which has the property that the restrictions of the trajectories in* $\mathcal{F}^\Delta$ *to* $\mathring{\Omega}^{ad} = \{(t,x) : h_\alpha(t,x) < 0\}$ *agree with the trajectories in the field* $\mathcal{F}$*. Thus the field* $\mathcal{F}^\Delta$ *extends the field* $\mathcal{F}$ *from int* Ω^{ad} *to the set* Ω_{ad}^Δ*.*

Essentially this holds since our assumptions on the reference trajectory guarantee that for Δ small enough the trajectories which follow the same switching structure as Γ, but obey the relaxed state constraint, are extremals for (Q_Δ).

Proof. We tacitly assume that $\Delta > 0$ is small enough. For the new problem (Q_Δ) the original reference trajectory $\Gamma = (x_*, u_*)$ is no longer extremal. The exit arc Γ_+ still is extremal, but now at time τ_2 the switching surface $\mathcal{S}$ is encountered without meeting the constraint and thus for (Q_Δ) the backward propagation of Γ_+ is the bang-bang trajectory corresponding to the trajectory of (Q) that has a junction point at $(\tau_2, x_*(\tau_2))$. Since M_α is an embedded submanifold which is transversal to $\mathcal{S}$, the switching surface $\mathcal{S}$ still intersects the slice $M_\alpha^\Delta = \{(t,x) : h_\alpha(t,x) = \Delta\}$ transversally. Thus again we integrate trajectories and the adjoint equation backward from time $\tau_2 + \varepsilon$ using the control $u \equiv b$ until the switching surface is encountered. All trajectories with terminal points in P at time $\tau_2 + \varepsilon$ will either hit $\mathcal{S}$ or the surface M_α^Δ and there exists an $(n-1)$-dimensional submanifold P_0^Δ of P with the property that trajectories emanating from P_0^Δ hit $\mathcal{S} \cap M_\alpha^\Delta$. Over some compact interval $[\tau_1 - \vartheta, \tau_2 + \varepsilon]$ the constraint M_α^Δ still is control invariant of relative degree 1. Also, by continuous dependence on initial conditions, it follows that the solutions to the differential equation that keep M_α^Δ invariant exist if P, ϑ, and ε are chosen small enough. We therefore can propagate trajectories onto M_α^Δ by integrating the boundary control (9) backward from points in $\mathcal{S} \cap M_\alpha^\Delta$. Assumptions $(C1)$-$(C3)$ will then be satisfied for these trajectories by simple continuity arguments and thus this defines a field of extremals on the constraint manifold M_α^Δ. Since ν_α is positive, integrating $u \equiv a$ backward from points on M_α^Δ again gives extremals of (Q_Δ) and these trajectories still cross the original constraint manifold M_α transversally. It follows from the uniqueness of solutions for the system $\dot{x} = f(t,x) + ag(t,x)$, $\dot{t} = 1$, that these trajectories must match up with the trajectories of the original field $\mathcal{F}$ as they encounter the original constraint M_α. Thus the two fields have the strong matching property postulated above. In fact, the two syntheses $\mathcal{F}$ and $\mathcal{F}^\Delta$ are homotopic. □

Proposition 6.2 *For $(t,x) \in \Omega_{ad}$ fixed, the value function $V^\Delta(t,x)$ of the local field $\mathcal{F}^\Delta$is continuous from the right at $\Delta = 0$,*

$$\lim_{\Delta \to 0+} V^\Delta(t,x) = V^{\mathcal{F}}(t,x).$$

Proof. As above, it follows from the stability properties of the fields $\mathcal{F}^\Delta$ in Proposition 6.1 that the trajectories $\xi^\Delta(t,x)$ (starting at x at time t) converge to the trajectory $\xi(t,x)$ of the field $\mathcal{F}$ uniformly on the full interval $[t, t_2 + \varepsilon]$ as $\Delta \to 0+$. This implies the result. □

7 Local Optimality of the Reference Trajectory

Theorem 7.1 *Assume conditions (A) - (E) hold. Then there exists a neighborhood Ω of the restriction Γ_ρ of the reference trajectory Γ_* to the interval $I = [\tau_1 - \vartheta, \tau_2 + \varepsilon]$ in $C(I)$ such that Γ_ρ is optimal for the problem $(\hat{Q})$ when compared with any other admissible trajectory which lies in Ω and has initial value $x_*(\tau_1 - \vartheta)$ at time $\tau_1 - \vartheta$. With V^* the value function of a local embedding of the remaining portion of the reference trajectory over $[t_2 + \varepsilon, T]$, the restriction of Γ_* to $[\tau_1 - \vartheta, T]$ is a strong local minimum.*

Proof. Let $\eta : [\tau_1 - \vartheta, \tau_2 + \varepsilon] \to [a,b]$ be an arbitrary admissible control for which the corresponding trajectory ξ with initial condition $x = x_*(\tau_1 - \vartheta)$ lies in Ω. By Proposition 6.2 it suffices to show that for all $\Delta > 0$ small enough we have $J(\eta) \geq V^\Delta(\tau_1 - \vartheta, x)$. For $\Delta > 0$ the trajectory ξ does not meet the constraint M_α^Δ, but entirely lies in the interior of the admissible state space. Without loss of generality we may therefore consider the case $\Delta = 0$ and assume that ξ does not meet the state space constraint.

However, we still need to handle the times when ξ crosses the surfaces $\mathcal{C}$ or $\mathcal{S}$ where V is not differentiable. For the local synthesis constructed above this argument is significantly simplified by the tight control that geometric properties give over the corresponding set of times. Indeed, there exist times α, β and γ, $\tau_1 - \vartheta < \alpha \leq \beta < \gamma < \tau_2 + \varepsilon$, such that the trajectory ξ lies in the open stratum R_3 for $t \in [\tau_1 - \vartheta, \alpha)$, then lies in the stratum $\mathcal{C}$ over a closed interval $[\alpha, \beta]$ (possibly $\alpha = \beta$) where it follows a trajectory of the field, then passes through the open stratum R_2 for $t \in (\beta, \gamma)$, crosses the switching surface $\mathcal{S}$ at time γ and then traverses the open stratum R_1 for $t \in (\gamma, \tau_2 + \varepsilon]$ (c.f. Fig. 2). The reason for this is the fact that the stratum $\mathcal{C}$, consisting of trajectories that only have contact points with the constraint M_α, is an integral manifold of the dynamics for the constant control $u = a$ in (t, x)-space. Thus if $\eta = a$ a.e. in some interval $[\alpha, \beta]$, then the trajectory ξ lies on the stratum $\mathcal{C}$ for this time interval, and if η is not equal to a a.e. over any small interval $(\beta, \beta + \omega)$, $\omega > 0$, then the trajectory ξ leaves $\mathcal{C}$ at time β and, since the integral manifolds corresponding to the control $u = a$ are disjoint, cannot return to $\mathcal{C}$ any more because a is the lowest value in the control set. Note that we do allow for the case that $\alpha = \beta$ when the trajectory ξ simply crosses $\mathcal{C}$. Similarly, it follows from assumption (E) that, no matter what the control η is, the trajectory ξ transversally crosses $\mathcal{S}$ at some time $\gamma > \beta$. The degenerate case $\beta = \gamma$ would arise when ξ crosses the switching surface through the stratum N_2 on the constraint, but this case is not possible if the trajectory ξ does not meet the constraint.

The value function is continuously differentiable on the open regions R_i, $i = 1, 2, 3$, and thus for times t in the intervals $(\tau_1 - \vartheta, \alpha)$, (β, γ) and $(\gamma, \tau_2 + \varepsilon)$ we have

$$\frac{d}{dt} V(t, \xi(t)) = V_t(t, \xi(t)) + V_x(t, \xi(t))(f(t, \xi(t)) + \eta(t) g(t, \xi(t))) \geq 0. \tag{50}$$

Integrating over any compact subinterval $[t_1, t_2]$ contained in these intervals thus gives

$$V(t_1, \xi(t_1)) \leq V(t_2, \xi(t_2)).$$

Since the value function V is continuous we therefore get

$$V(\tau_1 - \vartheta, x) \leq V(\alpha, \xi(\alpha))$$

and

$$V(\beta, \xi(\beta)) \leq V(\gamma, \xi(\gamma)) \leq V^*(\tau_2 + \varepsilon, \xi(\tau_2 + \varepsilon)) = J(\eta).$$

Although the value function V itself is not differentiable at points in the stratum $\mathcal{C}$, its restriction to this stratum is. In fact, since our problem formulation is given in Meyer form, the composite function $t \mapsto V(t, \xi(t))$ is constant over the interval $[\alpha, \beta]$ and thus we have

$$V(\alpha, \xi(\alpha)) = V(\beta, \xi(\beta)).$$

Hence overall $J(\eta) \geq V(\tau_1 - \vartheta, x)$. The fact that we can control the set of times along the interval $[\tau_1 - \vartheta, \tau_2 + \varepsilon]$ where the composition $t \mapsto V(t, \xi(t))$ is not differentiable to be finite allows to use the classical argument. □

8 Conclusion: Optimal Syntheses with Exit-Arcs on Switching Surfaces

The synthesis constructed in this paper occurs when the exit arc Γ_+ is embedded into a family of bang arcs corresponding to the same control. The key assumption corresponding to this is condition (D) making the connection with the adjoint variables. This situation is the codimension 0 scenario for problems of the type (Q) over a fixed time interval $[0, T]$ when no terminal constraints are imposed (and when no state constraints are active at the final time as it was considered here). In the presence of terminal constraints other scenarios arise also and we briefly indicate the changes (see Fig. 4). Consider a reference trajectory x_* corresponding to a reference control u_* that has two boundary segments over intervals $[\tau_1, \tau_2]$ and $[\rho_1, \rho_2]$ as shown in Fig.

4. The structure of the synthesis considered in this paper arises around the boundary arc over $[\tau_1, \tau_2]$, but the structure is different along the exit portion of the reference trajectory at time ρ_2. If we assume that conditions (A) - (C) hold at the exit junction $(\rho_2, x_*(\rho_2))$, it follows that the reference control u_* is discontinuous at the exit-junction and thus the associated switching function has a zero at ρ_2. Hence, as above, there exists a bang-bang trajectory which switches from say $u \equiv b$ to $u \equiv a$ at the exit-time ρ_2 (making a normalization which would be consistent with the one made above for the interval $[\tau_1, \tau_2]$). The graph of this trajectory only has a contact point with the lower constraint M_β at $(\rho_2, x_*(\rho_2))$ and this point will be part of another "switching surface" $\mathcal{R}$. However, the reference trajectory need not be transversal to $\mathcal{R}$, but may entirely lie in $\mathcal{R}$. The structure of the synthesis in this case is simpler than the one constructed in this paper in the sense that the local field is obtained by integrating the constant control $u \equiv b$ backward from both the constraint M_β and the switching surface $\mathcal{R}$. Note, however, that even in this case as the trajectories are propagated backward further, the type of synthesis analyzed in this paper arises for earlier boundary arcs like the one over the interval $[\tau_1, \tau_2]$. The more intricate aspect now is the analysis of this switching surface $\mathcal{R}$ which is made up of graphs of trajectories of the system and no longer is just a transit stratum like the switching surface $\mathcal{S}$ in the synthesis constructed in this paper. However, in this case the analysis is almost fully an analysis for the problem without state space constraints. Once the structure of the synthesis on $\mathcal{R}$ is understood for the problem without constraints, it seems rather straightforward to impose the state space constraint, at least under nondegenerate assumptions. Certainly this is the case in low dimensions, but it requires a more precise formulation and analysis for problems in higher dimensions. But under assumptions (A)-(C) the intersection $\mathcal{R} \cap M_\beta$ does not support graphs of trajectories and thus the graphs of all trajectories in $\mathcal{R}$ are transversal to M_β. The mathematical reasoning developed in this paper therefore generalizes to handle this second setting as well.

In this sense our construction provides a stepping stone to an analysis of these and other more complicated situations. It also becomes important for the inclusion of *singular interior arcs* into our construction. This is an important and common structure which is not covered by the geometric arguments in this paper, but does arise frequently. For example, problems of minimizing the base transit times in semiconductor devices [30] are best formulated as single-input optimal control problems with state space constraints which are just hard limits on some of the states and all have order 1 [25]. For this problem optimal solutions always have boundary arcs, but from one side these boundary arcs are joined by a singular arc, not a bang arc as we considered in this paper. In dimension 3 the optimal synthesis near a junction between an interior singular arc and a boundary arc resembles the local synthesis of Fig. 4 at the junction point $(\rho_2, x_*(\rho_2))$, except that the singular arc is approached by the bang arcs from either side. This construction is outlined in [26] and the mathematical analysis is almost identical in the cases when the switching surface $\mathcal{S}$ consists of graphs of bang-bang trajectories or of a singular surface, but in case of a singular surface the geometric structure itself is only generic in low dimensions. In order to limit the length of this paper we only considered one specific scenario, but carried out the construction in detail.

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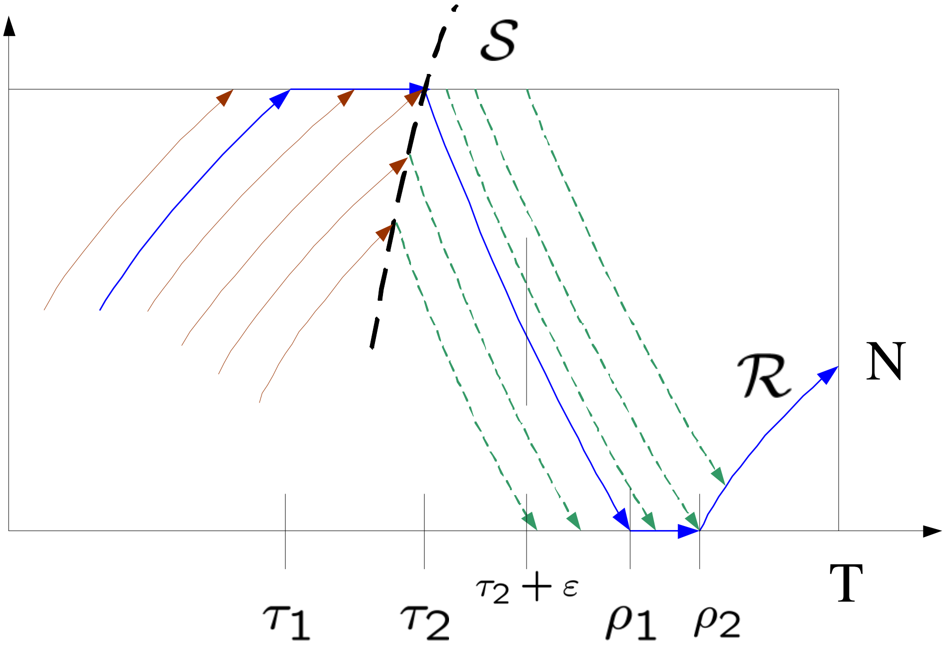

Figure 4: Local synthesis with terminal constraints

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