computations needed to whiten or to invert the noise covariance matrix for a nonredundant system increases as $\sim M^3/3$ and all elements of A, C, $\bar{y}$, and W have to be saved.

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A Mapping Result Between Wiener Theory and Kalman Filtering for Nonstationary Processes

AYRE NEHORAI AND MARTIN MORF

Abstract—**A connection between Wiener theory and Kalman filtering is presented for discrete nonstationary finite-dimensional processes. A time-varying realization of input–output relation by matrix fraction description (MFD) form and the observer state-space form in operator notation are used to derive the innovation filters of the nonstationary processes and to express the Kalman gain in terms of Wiener theory. The results generalize the stationary case.**

I. Introduction

The major contribution of Wiener [1] to estimation theory was the derivation of the optimal filter and predictor for stochastic stationary scalar processes using spectral factorization techniques. Kalman [2] replaced the input–output signal model of Wiener by a state-space model, and obtained a recursive solution for the optimal state estimator for stationary and nonstationary cases.

The main difficulty in relating the results of Wiener to Kalman filtering for nonstationary signals is due to the fact that the usual analysis of stationary signals by frequency domain methods (Laplace or z-transforms) is not applicable. We use here time domain analysis by "matrix-maps" (see, e.g., [3]–[6]) to obtain a connection between Wiener theory and Kalman filtering for discrete nonstationary finite-dimensional processes. In this approach the signals are described in the time domain over the total observation interval as a single object.

Section II discusses alternative models of finite-dimensional time-varying discrete linear systems. The first model is of input–output relationship by a banded matrix fraction description (MFD) of the system operator; the second is a time-varying observer state-space model in operator notation. The relationship between these models can be found using the results of [7]. In Section III we use the alternative system models to derive two expressions for the innovation filter of the observed nonstationary process, one by Wiener theory and the other by Kalman filter. The relationship of [7] enables us to connect the two results and express the Kalman gain in terms of the innovation filter or covariance factor by Wiener theory. The results generalize the stationary case appearing in [8]–[10].

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II. Alternative Models of Time-Varying Finite-Dimensional Linear Systems

Input–Output Model

An m-input p-output discrete time-varying process of order q plus measurement noise v_i, can be modeled by

$$\begin{cases} z_i = -\sum_{j=i-q}^{i-1} A_{i,j} z_j + \sum_{j=i-q}^{i-1} B_{i,j} u_j \\ y_i = z_i + v_i \end{cases} \tag{1}$$

where the coefficients $A_{i,j}$ and $B_{i,j}$ are $p \times p$ and $p \times m$ dimensional matrices, respectively. Equation (1) is similar to the scalar one used by Whittle [11]. Assuming that $u_i = 0$ for $i < 0$, it can be expressed globally in time by the MFD model

$$\mathcal{Y}_{0:T} = \boldsymbol{A}^{-1}\boldsymbol{B}\mathcal{U}_{0:T} + \mathcal{V}_{0:T} \tag{2}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are, respectively, lower and strict lower block banded matrices, and

$$\mathcal{Y}_{0:T} = [y_0^T, \; y_1^T, \; \cdots, \; y_T^T]^T \tag{3}$$

and similarly for $\mathcal{U}_{0:T}$ and $\mathcal{V}_{0:T}$. The structure of $\boldsymbol{A}$ and $\boldsymbol{B}$ should be noted. The lower block triangular form corresponds to the causality assumption. The block banded form is related to the recursion (1). These matrices are special cases of "matrix-maps" which are global linear operator representations of a system impulse response.

Global State-Space Model

A finite-dimensional time-varying linear system can be described also by the (pointwise) state-space model

$$\begin{cases} x_{i+1} = \phi_{i+1,i} x_i + \Gamma_{i+1,i} u_i \\ y_i = H_i x_i + v_i \end{cases} \qquad i \geq 0 \tag{4}$$

where the state x_i is an $n \times 1$ vector. Stacking these equations together we have

$$\begin{cases} \mathcal{X}_{0:T} = \Phi \mathcal{X}_{0:T} + \Gamma \mathcal{U}_{0:T} + [I_n 0, \; \cdots, \; 0]^T x_0 \\ \mathcal{Y}_{0:T} = \boldsymbol{H} \mathcal{X}_{0:T} + \mathcal{V}_{0:T} \end{cases} \tag{5}$$

where

$$\Phi = \begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & \phi_{i,i-1} & & \ddots \\ & & & 0 \end{bmatrix} \tag{6a}$$

$$\Gamma = \begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & \Gamma_{i,i-1} & & \ddots \\ & & & 0 \end{bmatrix} \tag{6b}$$

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$$H = \begin{bmatrix} H_0 & & 0 \\ & \ddots & \\ 0 & & H_T \end{bmatrix}.$$

The Observer State-Space Realization

To realize the noise-free process $\mathcal{Z}_{0:T} = A^{-1}B\mathcal{U}_{0:T}$ we shall use the operator state-space model (4)–(6), with $x_0 = 0$ and the special block observer form

$$\phi_{oi+1,i} = \begin{bmatrix} -A_{i+1,i} & I_p & & 0 \\ \cdot & & \ddots & \\ \cdot & & & \\ \cdot & 0 & & I_p \\ -A_{i+q,i} & & & 0 \end{bmatrix} \quad \Gamma_{oi+1,i} = \begin{bmatrix} B_{i+1,i} \\ \cdot \\ \cdot \\ \cdot \\ B_{i+q,i} \end{bmatrix} \tag{7a}$$

$$H_{oi} = [I_p 0, \cdots, 0]. \tag{7b}$$

Note that in this notation matrices as $A_{T+1,T-1}$, etc. (i.e., with first subindex larger than T) are needed for (6a) and (6b). These matrices can be arbitrarily chosen since they do not show up in the relations to the input–output model.

In [7] we investigate some other realizations and connections between input–output MFD forms and state-space realizations of time-varying processes. Here we note only one result from [7] for the observer form, which will be used later

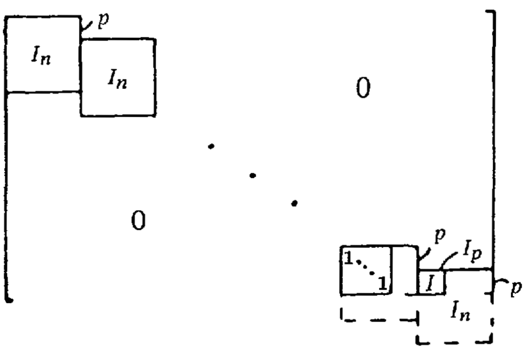
$$H_0(I-\Phi_0)^{-1} = A^{-1}\Psi \tag{8a}$$

and

$$\Psi\Gamma = B \tag{8b}$$

where

$\Psi =$



$$p(T+1) \times n(T+1). \tag{9}$$

This matrix Ψ is lower block banded. Each of its identity block entries I_n is shifted with respect to the other by p rows. (Note that here $n = pq$.) In the lower right corner the lower parts of these block entries are being chopped so as to yield the desired number of rows, namely $p(T + 1)$. The relationship (8a) generalized the time-constant case of [8, equation (18)] and [12, sect. 6.4, equation (38)]. It may be noted that a good way to check (8a) is by direct verification of $AH_0 = \Psi(I - \Phi_0)$. Also observe that $\mathcal{Z}_{j:T} = [H_0(I - \Phi_0)^{-1}]_{j:T,j:T}[I_n 0, \cdots, 0]^T x_j$ for $\mathcal{U}_{j:T} = 0$.

III. The Innovation Filters and the Mapping Result

To derive the alternative innovation filter expressions of y_i by Wiener theory and by Kalman filtering, assume that the statistical properties of the signals are known and given by $Eu_iu_j^T = Q_i\delta_{ij}$, $Eu_iv_j^T = 0$, $Ev_iv_j^T = R_i\delta_{ij}$, $Q_i \geq 0$, $R_i > 0$, where E is the expectation operator and δ_{ij} is the Kronecker delta. In global form $E\mathcal{U}_{0:T}\mathcal{U}_{0:T}^T \triangleq Q =$ block diag $\{Q_i\}$, $E\mathcal{U}_{0:T}\mathcal{V}_{0:T}^T = 0$, $E\mathcal{V}_{0:T}\mathcal{V}_{0:T}^3 \triangleq R_v =$ block diag $\{R_i\}$. For the state-space model (4) we also assume that $Ex_0 = 0$, $Ex_0x_0^T = \pi_0\, Eu_ix_0^T = 0$.

The Innovation Filter by Wiener Theory

From Wiener theory, the causal innovation filter operator of the (nonstationary) measurement signal y_i, is given by the causal square root or Cholesky factor of the covariance matrix R_y, in input–output form. (Note that other time domain Wiener filters of nonstationary processes were used, e.g., in [4]–[6], [13], and [14]. The main difference is in our use of MFD models along with their connections to state-space models.) Assuming that y_i is a finite order process plus the noise v_i and that the initial conditions are zero, we can associate with R_y the input–output relation (1), thus

$$\begin{aligned} R_y &= A^{-1}BQB^TA^{-T} + R_v \\ &= A^{-1}(BQB^T + AR_vA^T)A^{-T}. \end{aligned} \tag{10}$$

Denoting the lower block triangular Cholesky factor of $(BQB^T + AR_vA^T)$ by $\tilde{C}$, we get the following block triangular decomposition of R_y:

$$R_y = A^{-1}\tilde{C}\tilde{C}^TA^{-T}. \tag{11}$$

This implies that the innovation filter operator of the measurement signal y_i by Wiener theory is given by the MFD form

$$R_y^{1/2} = A^{-1}\tilde{C}. \tag{12}$$

The Innovation Filter by Kalman Filtering

The Kalman filter equations for time-varying models under the above assumptions are

$$\hat{x}_{i+1|i} = \phi_{i-1,i}\hat{x}_{i|i-1} + K_{gi+1,i}\epsilon_i \qquad \hat{x}_{0,-1} = 0 \tag{13a}$$

$$\epsilon_i = y_i - H_i\hat{x}_{i|i-1} \tag{13b}$$

$$R_i^\epsilon = E\epsilon_i\epsilon_i^T = H_iP_{i|i-1}H_i^T + R_i \tag{13c}$$

$$K_{gi+1,i} = \phi_{i+1,i}\, P_{i|i-1}H_i^TR_i^{-\epsilon} \tag{13d}$$

$$P_{i+1|i} = \phi_{i+1|i}\, P_{i|i-1}\phi_{i+1|i}^T + \Gamma_{i-1,i}Q_i\Gamma_{i+1,i}^T - K_{gi+1,i}\, R_i^\epsilon K_{gi+1,i}^T \tag{13e}$$

$$P_{0,-1} = \pi_0.$$

From the first two Kalman equations one obtains

$$\begin{cases} \hat{x}_{0:T} = \Phi\hat{x}_{0:T} + K_g\epsilon_{0:T} \\ \epsilon_{0:T} = \mathcal{Y}_{0:T} - H\hat{x}_{0:T} \end{cases} \tag{14}$$

where

$$K_g = \begin{bmatrix} 0 & & & 0 \\ \ddots & \ddots & & \\ 0 & K_{gi,i-1} & \ddots & \\ & & \ddots & 0 \end{bmatrix}. \tag{14a}$$

These equations yield the innovation representation

$$\mathcal{Y}_{0:T} = [I + H(I-\Phi)^{-1}K_g]\epsilon_{0:T} \tag{15}$$

From (15) we obtain the innovation filter operator by Kalman filtering

$$R_y^{1/2} = [I + H(I-\Phi)^{-1}K_g]R_\epsilon^{1/2} \tag{16}$$

where $R_\epsilon^{1/2}$ is the Cholesky factor of the (block diagonal) matrix R_ϵ. Now substituting the relation (8a) into (16), we have

$$R_y^{1/2} = (I + A^{-1}\Psi K_g) R_\epsilon^{1/2}. \tag{17}$$

The matrix Ψ acts as a shift operator on the block columns of K_g, so that ΨK_g is a lower block banded matrix, cf. (8b). Denote

$$K = \Psi K_g \tag{18}$$

then K is lower block banded and

$$R_y^{1/2} = (I + A^{-1}K) R_\epsilon^{1/2}. \tag{19}$$

The Mapping Result

Comparing now the two results (12) and (19) for the covariance factor $R_y^{1/2}$, we find

$$(I + A^{-1}K) R_\epsilon^{1/2} = A^{-1}\tilde{C}. \tag{20}$$

Let

$$C = \tilde{C} R_\epsilon^{-1/2} \tag{21}$$

then we conclude

$$K = C - A. \tag{22}$$

The global relation (22) can be reduced to a local one by noting that the matrices K, C, and A are all lower block banded, hence

$$K_i = C_i - A_i \tag{23}$$

where K_i (respectively, C_i, A_i) is the ith block column of K (respectively, C, A). Now notice from (18) that K_i corresponds to the Kalman gain $K_{gi+1,i}$, so that we have obtained an alternative expression for the Kalman gains at different times in terms of Wiener theory. This result generalizes the stationary one in [8, p. 662], [9, p. 173], and [10, ch. 7].

We should note that the factorization of a block matrix is not unique, and that (22) is true under the *block* factorization (11), giving the block banded C necessary for a recursive realization of (12). Also notice that the mapping result (23) does not show yet how one can get the matrix $R_\epsilon^{1/2}$ from the multivariable Wiener theory. However, by noting from (14c) and (18) that K has to be strictly lower banded, it is clear that $R_{\epsilon i,i}^{1/2} = C_{i,i}^{-1} = \tilde{C}_{i,i}$.

IV. Conclusion

A relationship between Wiener theory and Kalman filtering has been presented for nonstationary finite-dimensional processes. The results give an expression for the Kalman gain in terms of the innovation filter by Wiener theory, which generalizes the stationary case. Similar relations can be found for the filter and smoothing estimators.

The main difference between the modes of this paper and others which used matrix operators to model time varying processes, e.g., in [4]–[6] and [13], [14], is our use of system operators in MFD forms along with their connection to state-space forms. Several additional general results for time-varying realizations of multivariable linear systems are given in [7]. Here we note that among other possible realizations of the system operator matrix, the one by diagonals turns out to be intimately related to the ladder form. The advantages of this form, including modular structure and good numerical behavior makes it even more attractive for future research.

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A Simulation Aid to Gain Estimates for Robust Tuning Regulators

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Abstract—**Gain estimates for Davison's robust tuning regulator are shown to be directly computable from plant open-loop step data using low-order simulation techniques. At no stage of the procedure is a detailed model of plant dynamics required.**

I. Introduction

The robust tuning regulator originally proposed by Davison [1] provides a simple but effective way of generating integrating process controllers without the need to have a detailed model of the plant available. More precisely, given the m-input/m-output *stable* multivariable plant described by the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{1}$$

in R^n, the integrating unity negative output feedback controller with transfer function matrix

$$K(s) = G^{-1}(o)\frac{\epsilon}{s} \tag{2}$$

will stabilize the plant, reject constant disturbances, and track step set-point changes for gains ϵ in some (nonempty) range $o < \epsilon < \epsilon^*$ provided that $G(o)$ is nonsingular. The matrix $G(o)$ can be computed directly from the plant transfer function matrix $G(s) = C(sI - A)^{-1}B$ (if available) or

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