Effects of Sensor Placement on Acoustic Vector-Sensor Array Performance

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***Abstract*—We consider the role played by the sensor locations in the optimal performance of an array of acoustic vector sensors. First we derive an expression for the Cramér–Rao bound on the azimuth and elevation of a single far-field source for an arbitrary acoustic vector-sensor array in a homogeneous wholespace and show that it has a block diagonal structure, i.e., the source location parameters are uncoupled from the signal and noise strength parameters. We then derive a set of necessary and sufficient geometrical constraints for the two direction parameters, azimuth and elevation, to be uncoupled from each other. Ensuring that these parameters are uncoupled minimizes the bound and means they are the natural or "canonical" location parameters for the model. We argue that it provides a compelling array design criterion. We also consider a bound on the mean-square angular error and its asymptotic normalization, which are useful measures in three-dimensional bearing estimation problems. We derive an expression for this bound and discuss it in terms of the sensors' locations. We then show that our previously derived geometrical conditions are also sufficient to ensure that this bound is independent of azimuth. Finally, we extend those conditions to obtain a set of geometrical constraints that ensure the optimal performance is isotropic.**

***Index Terms*— Array geometry, direction estimation, vector sensors, velocity sensors.**

I. Introduction

The passive direction-of-arrival (DOA) estimation problem, in which the bearings of a number of far-field acoustic sources are determined, is of great importance in many underwater applications. The traditional solution is to use a spatially distributed array of omnidirectional pressure sensors, and many estimation techniques have evolved for this scenario. As the demand for smaller arrays that perform better at a lower signal-to-noise ratio (SNR) has increased, the idea of measuring particle velocity, as well as pressure, has arisen [1], [2]. This has coincided with a surge of interest in particle velocity sensors and improvements in fabrication techniques [3] to make vector-sensor arrays a practical reality [4].

An acoustic vector sensor measures the acoustic pressure and all three components of the acoustic particle velocity at a single point in space. Thus, while standard pressure sensors can only utilize the directional information present in the propagation delays between sensors, each vector sensor can extract further directional information directly from the structure of the velocity field. This directional information permits, for example, a single vector sensor to identify up to two sources [5]. By making use of this extra information, arrays of vector sensors are able to improve source localization accuracy without increasing array aperture.

Both theoretical and practical work has been published on the topic of acoustic vector sensors. A model for acoustic vector sensors and a simple DOA estimation algorithm for a single sensor were introduced in [1], [2], preliminary theoretical performance analyses appear in [1], [2], [6], and [7], and an in-depth analysis of the Cramér–Rao bound (CRB) for a single source appears in [8]. Their use has also been examined on or near surfaces [9], and ESPRIT and Root-MUSIC algorithms have recently been applied to arrays of velocity-sensor triads [10], [11]. Meanwhile, acoustic vector sensors have been constructed [12] and linear arrays of them built and subjected to sea trials, [4], [13]–[15].

In this paper, we provide a derivation of the CRB, which is a lower bound on the estimation error of the DOA parameters that is achieved asymptotically by the maximum-likelihood (ML) estimator, for a single source impinging on an arbitrary array of acoustic vector sensors. This is the first open-literature derivation of this result, which is analyzed in considerable detail in [8]. Since vector sensors are inherently three-dimensional (3-D), a source must be parameterized by both its azimuth and elevation in order to completely specify the measurement model. The CRB on the direction parameters is thus a matrix of order two that lower bounds the 2×2 covariance matrix of an estimator of both angles, in the sense that their difference is positive definite.

Since the CRB is independent of any particular estimator and gives the asymptotic performance of the ML estimator (which is asymptotically optimal in the absence of prior information), it forms a very useful criterion for array design. Once we have derived the bound, we consider how the array's geometry (i.e., the sensor locations) affects the bound. In particular, we derive a set of necessary and sufficient constraints on the sensor positions for the azimuth and elevation to be uncoupled in the bound. This means that accuracy with which one of the arrival angles can be estimated is independent of knowledge of the other. In addition, as explained later, ensuring the parameters are uncoupled tends to minimize the bound and hence maximize the potential estimation accuracy.

Manuscript received November 8, 1997; revised November 16, 1998. This work was supported by the Air Force Office of Scientific Research under Grant F49620-99-1-0067 and Grant F49620-97-1-0481, by the National Science Foundation under Grant MIP-9615590, and by the Office of Naval Research under Grant N00014-98-1-0542.

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Publisher Item Identifier S 0364-9059(99)01229-7.

The result we obtain was shown to be sufficient for pressure-sensor arrays in [16]. We show that those same conditions are in fact necessary and also apply to vector-sensor arrays. We also show that the direction parameters are uncoupled from the signal and noise powers, indicating that optimal direction estimation is unaffected by lack of knowledge of the signal and noise strengths.

Estimating both azimuth and elevation is equivalent to estimating a 3-D unit-length bearing vector. We consider the mean-square angular error (MSAE) of an estimator of such a vector and its asymptotic normalized value (defined in [17]). Both are useful performance measures that are independent of the chosen coordinate system. We find an expression for a lower bound on these quantities that is attained asymptotically by the ML estimator. We discuss the structure of the bound in terms of the array geometry and show that the above conditions for an uncoupled CRB are also sufficient to ensure that the MSAE bound is independent of azimuth. We also extend those conditions to find a sufficient set of geometric constraints that ensure it is completely independent of the DOA, i.e., that give the array an optimal isotropic performance.

The structure of the paper is as follows. In Section II, we explain the measurement model for an array of acoustic vector sensors and introduce our statistical assumptions. In Section III, we give the derivation of the CRB and show that the direction parameters are uncoupled from the signal and noise powers. In Section IV, we derive the necessary and sufficient conditions on sensor locations for the uncoupling of azimuth and elevation in the bound. We also derive the bounds on the mean-square angular error and its asymptotic normalization and obtain the two sets of sufficient geometric conditions that ensure that these bounds are either independent of azimuth or isotropic. Section V concludes the paper.

II. Measurement Model

A. Array Output

We consider a narrow-band plane wave traveling in an isotropic, quiescent, homogeneous fluid wholespace, and impinging on an array of m acoustic vector sensors located at $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_m$ (see [2] for details). The plane wave is parameterized by its azimuth $\phi \in [0, 2\pi)$ and elevation $\psi \in [-\pi/2, \pi/2]$, and the problem is to determine the DOA parameter $\boldsymbol{\theta} = [\phi, \psi]^T \in \boldsymbol{\Theta} \triangleq [0, 2\pi) \times [-\pi/2, \pi/2]$.

In [2] and [8], we derive the measurement model for n sources impinging on such an array. For a single source, the measurement vector is a $4m$-element complex vector $\boldsymbol{y}(t) = [y_{p1}(t), \boldsymbol{y}_{v1}^T(t), \cdots, y_{pm}(t), \boldsymbol{y}_{vm}^T(t)]^T$ and is given by

$$\boldsymbol{y}(t) = \boldsymbol{a}(\boldsymbol{\theta})p(t) + \boldsymbol{e}(t) \tag{1}$$

where $y_{pk}(t)$ and $\boldsymbol{y}_{vk}$ are the (scalar) pressure measurement and (vector) velocity measurement made by the kth sensor at time t, $p(t)$ is the pressure complex envelope of the source signal, and $\boldsymbol{e}(t)$ represents noise. The array's steering vector $\boldsymbol{a}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{a}(\boldsymbol{\theta}) = \boldsymbol{a}_{\mathrm{P}}(\boldsymbol{\theta}) \otimes \boldsymbol{h}(\boldsymbol{\theta}) \tag{2}$$

where $\boldsymbol{a}_{\mathrm{P}}(\boldsymbol{\theta})$ is the steering vector of an equivalent pressure-sensor array, i.e., an array with a single pressure sensor at each of the m locations $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_m$, while $\boldsymbol{h}(\boldsymbol{\theta})$ is the response of a single vector sensor located at the origin and $\otimes$ is the Kronecker product. If $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_m$ are in units of wavelengths corresponding to the source's center frequency, we have

$$\boldsymbol{a}_{\mathrm{P}}(\boldsymbol{\theta}) = [e^{i2\pi(\boldsymbol{r}_1^T\boldsymbol{u})}, \cdots, e^{i2\pi(\boldsymbol{r}_m^T\boldsymbol{u})}]^T \tag{3}$$

$$\boldsymbol{h}(\boldsymbol{\theta}) = [1, \boldsymbol{u}(\boldsymbol{\theta})^T]^T \tag{4}$$

where $\boldsymbol{u}(\boldsymbol{\theta}) = [\cos\phi\cos\psi, \sin\phi\cos\psi, \sin\psi]^T$ is the source's bearing vector (i.e., the unit length vector pointing from the origin toward the source).

Practical acoustic vector sensors consist of a co-located pressure sensor (hydrophone) and triad of orthogonal velocity sensors (geophones) [4], [12]. In the derivation of the measurement model, it is assumed that the output of each geophone is proportional to the cosine of the angle between $\boldsymbol{u}$ and one of the coordinate axes. This assumption requires that: 1) the geophones have a cosine response and 2) each of the m vector sensors in the array has a known orientation and that this orientation is used to rotate the data so as to align them with the coordinate axes. The speed of sound in the medium and the ambient density are also assumed to be known.

Note that $\boldsymbol{a}_{\mathrm{P}}(\boldsymbol{\theta})$ contains the phase delay information between noncoincident sensors and depends only on the sensor locations. On the other hand, $\boldsymbol{h}(\boldsymbol{\theta})$ accounts for the directional response of each component: omnidirectional for the pressure sensor and cosine for the velocity sensors, and is *independent* of the position of each sensor. It therefore conveys the DOA information present in the structure of the velocity field. It is the assumption that all vector sensors have a common alignment with the coordinate axes (by a rotation of the data) that permits the above Kronecker product expression for the steering vector. This, in turn, allows us to obtain considerably simpler CRB expressions than would otherwise be the case. Note that the usual pressure-sensor array model is a special case of this model obtained by setting $\boldsymbol{h}(\boldsymbol{\theta}) = [1, 0, 0, 0]^T$.

B. Statistical Assumptions

We assume that both the signal $p(t)$ and the noise $\boldsymbol{e}(t)$ are independent identically distributed (i.i.d.) zero-mean complex Gaussian processes. In addition, we assume that $p(t)$ and $\boldsymbol{e}(s)$ are independent for all s and t. These two processes are completely characterized by their covariance matrices

$$\mathrm{E}\{p(t)p^H(t)\} = \sigma_s^2 \tag{5}$$

$$\mathrm{E}\{\boldsymbol{e}(t)\boldsymbol{e}^H(t)\} = I_m \otimes \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_v^2 I_3 \end{bmatrix} \tag{6}$$

where the superscript ^{H} represents conjugate transposition, σ_s^2 is the signal power, σ_p^2 and σ_v^2 are the pressure-sensor and velocity-sensor noise variances, respectively, and I_m is the mth-order identity matrix. The assumed structure of the noise covariance is consistent with internal sensor noise. We allow the noise power to differ between pressure and velocity sensors to account for their different construction and the latter's directional sensitivity. With these assumptions, the data is also an i.i.d. zero-mean complex Gaussian process, with covariance matrix

$$R = \sigma_s^2 \boldsymbol{a}(\theta)\boldsymbol{a}^H(\boldsymbol{\theta}) + I_m \otimes \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_v^2 I_3 \end{bmatrix}. \tag{7}$$

There are a total of five unknowns: the two DOA parameters in $\boldsymbol{\theta}$, the signal power σ_s^2, and the noise variances σ_p^2 and σ_v^2.

III. Optimum Performance

We now derive an expression for the CRB on the DOA parameters of a single source using a vector-sensor array. The model and assumptions are those used in [2]. Using the results of [18], [2, Theorem 3.1] obtains a closed form expression for the *concentrated* CRB on the DOA parameters of n sources, i.e., it only requires inversion of a matrix of order $2n$ (the number of location parameters) instead of the whole Fisher information matrix (FIM). However, the theorem assumes that the noise covariance is proportional to an identity matrix. This is not so in our case, so we temporarily assume that $\eta \triangleq \sigma_v^2/\sigma_p^2$ is known. We normalize the data by a matrix Γ, i.e., $\tilde{\boldsymbol{y}}(t) = \Gamma \boldsymbol{y}(t)$, where

$$\Gamma = I_m \otimes \begin{bmatrix} \eta^{1/2} & 0 \\ 0 & I_3 \end{bmatrix} \tag{8}$$

and where we naturally choose the positive root of η. The transformed data may be written

$$\tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{a}}(\boldsymbol{\theta})p(t) + \tilde{e}(t). \tag{9}$$

This satisfies the assumptions of the theorem: the transformed noise covariance matrix is $\sigma_v^2 I_{4m}$. The transformed steering vector is still a Kronecker product $\tilde{\boldsymbol{a}}(\boldsymbol{\theta}) = \boldsymbol{a}_{\mathrm{P}}(\boldsymbol{\theta}) \otimes \tilde{\boldsymbol{h}}(\boldsymbol{\theta})$, where $\boldsymbol{a}_{\mathrm{P}}(\boldsymbol{\theta})$ is still given by (2), but now the orientational steering vector is $\tilde{\boldsymbol{h}}(\boldsymbol{\theta}) = [\eta^{1/2}, \boldsymbol{u}(\boldsymbol{\theta})]^T$. In the following, for compactness of notation, we suppress the explicit dependence of vectors on the DOA $\boldsymbol{\theta}$.

With the above transformation, the expression of [2, Theorem 3.1] for a single source becomes

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{\sigma_v^2}{2N} \frac{(|\tilde{\boldsymbol{a}}|^2\sigma_s^2 + \sigma_v^2)}{|\tilde{\boldsymbol{a}}|^2\sigma_s^4} \{\mathrm{Re}[(D^H \Pi_c D)^T]\}^{-1} \tag{10}$$

where $\Pi_c = I_{4m} - \tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^H/|\tilde{\boldsymbol{a}}|^2$ and $D = [\partial\tilde{\boldsymbol{a}}/\partial\phi, \partial\tilde{\boldsymbol{a}}/\partial\psi]$. Now, the entries of D may be written

$$\frac{\partial \tilde{\boldsymbol{a}}}{\partial \phi} = \frac{\partial \boldsymbol{a}_{\mathrm{P}}}{\partial \phi} \otimes \tilde{\boldsymbol{h}} + \boldsymbol{a}_{\mathrm{P}} \otimes \frac{\partial \tilde{\boldsymbol{h}}}{\partial \phi} \tag{11}$$

with a similar expression for $\partial\tilde{\boldsymbol{a}}/\partial\psi$. Since the entries of $\boldsymbol{a}_{\mathrm{P}}$ are exponential, its partial derivatives may be written

$$\partial \boldsymbol{a}_{\mathrm{P}}/\partial\phi = i2\pi\cos\psi[\boldsymbol{r}_1, \cdots, \boldsymbol{r}_m]^T \boldsymbol{v}_\phi \odot \boldsymbol{a}_{\mathrm{P}}, \tag{12}$$
$$\partial \boldsymbol{a}_{\mathrm{P}}/\partial\psi = i2\pi[\boldsymbol{r}_1, \cdots, \boldsymbol{r}_m]^T \boldsymbol{v}_\psi \odot \boldsymbol{a}_{\mathrm{P}} \tag{13}$$

where $\odot$ represents the elementwise product and the vectors $\boldsymbol{v}_\phi$ and $\boldsymbol{v}_\psi$, defined by $\boldsymbol{v}_\phi \cos\psi = \partial\boldsymbol{u}/\partial\phi$ and $\boldsymbol{v}_\psi = \partial\boldsymbol{u}/\partial\psi$, are given by

$$\boldsymbol{v}_\phi = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_\psi = \begin{bmatrix} -\cos\phi\sin\psi \\ -\sin\phi\sin\psi \\ \cos\psi \end{bmatrix} \tag{14}$$

respectively.

Now $\partial\tilde{\boldsymbol{h}}/\partial\phi = [0, \boldsymbol{v}_\phi^T \cos\psi]^T$ and $\partial\tilde{\boldsymbol{h}}/\partial\psi = [0, \boldsymbol{v}_\psi^T]^T$ and, since $(\boldsymbol{u}, \boldsymbol{v}_\phi, \boldsymbol{v}_\psi)$ is a right orthonormal triad,

$$\frac{\partial\tilde{\boldsymbol{h}}}{\partial\phi}^H \tilde{\boldsymbol{h}} = \frac{\partial\tilde{\boldsymbol{h}}}{\partial\psi}^H \tilde{\boldsymbol{h}} = \frac{\partial\tilde{\boldsymbol{h}}}{\partial\phi}^H \frac{\partial\tilde{\boldsymbol{h}}}{\partial\psi} = 0. \tag{15}$$

Without loss of generality, we choose the origin of our coordinate system to be the array centroid. It is then clear from (12) and (13) that

$$\frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi}^H \boldsymbol{a}_{\mathrm{P}} = \frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\psi}^H \boldsymbol{a}_{\mathrm{P}} = 0. \tag{16}$$

From these two orthogonality identities (15) and (16), it follows that

$$\frac{\partial\tilde{\boldsymbol{a}}}{\partial\phi}^H \tilde{\boldsymbol{a}} = \frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi} \boldsymbol{a}_{\mathrm{P}}|\tilde{\boldsymbol{h}}|^2 + |\boldsymbol{a}_{\mathrm{P}}|^2 \frac{\partial\tilde{\boldsymbol{h}}}{\partial\phi} \tilde{\boldsymbol{h}} = 0$$
$$\frac{\partial\tilde{\boldsymbol{a}}}{\partial\psi}^H \tilde{\boldsymbol{a}} = \frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\psi} \boldsymbol{a}_{\mathrm{P}}|\tilde{\boldsymbol{h}}|^2 + |\boldsymbol{a}_{\mathrm{P}}|^2 \frac{\partial\tilde{\boldsymbol{h}}}{\partial\psi} \tilde{\boldsymbol{h}} = 0. \tag{17}$$

Using these results, we calculate the entries of

$$D^H \Pi_c D = D^H D - D^H \tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^H D/|\tilde{\boldsymbol{a}}|^2. \tag{18}$$

It follows immediately from (17) that the second term of (18) is zero. Using the orthogonality of $\tilde{\boldsymbol{h}}$ and its derivatives (15), we find that the entries of the Hermitian matrix $D^H D$ are

$$\left|\frac{\partial\tilde{\boldsymbol{a}}}{\partial\phi}\right|^2 = \left|\frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi}\right|^2 |\tilde{\boldsymbol{h}}|^2 + |\boldsymbol{a}_{\mathrm{P}}|^2 \left|\frac{\partial\tilde{\boldsymbol{h}}}{\partial\phi}\right|^2$$
$$= (4\pi^2(\eta+1)S_{\phi\phi} + m)\ \cos^2\psi \tag{19}$$
$$\left|\frac{\partial\tilde{\boldsymbol{a}}}{\partial\psi}\right|^2 = \left|\frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\psi}\right|^2 |\tilde{\boldsymbol{h}}|^2 + |\boldsymbol{a}_{\mathrm{P}}|^2 \left|\frac{\partial\tilde{\boldsymbol{h}}}{\partial\psi}\right|^2$$
$$= 4\pi^2(\eta+1)S_{\psi\psi} + m \tag{20}$$
$$\frac{\partial\tilde{\boldsymbol{a}}}{\partial\phi}^H \frac{\partial\tilde{\boldsymbol{a}}}{\partial\psi} = \frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi}^H \frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\psi} |\tilde{\boldsymbol{h}}|^2$$
$$= 4\pi^2(\eta+1)S_{\phi\psi}\cos\psi \tag{21}$$

where we have defined

$$S_{\phi\phi} \triangleq \sum_j (\boldsymbol{r}_j^T \boldsymbol{v}_\phi)^2$$
$$S_{\psi\psi} \triangleq \sum_j (\boldsymbol{r}_j^T \boldsymbol{v}_\psi)^2$$
$$S_{\phi\psi} \triangleq \sum_j (\boldsymbol{r}_j^T \boldsymbol{v}_\phi)(\boldsymbol{r}_j^T \boldsymbol{v}_\psi). \tag{22}$$

So, in our single source case, $D^H D$ is in fact a real symmetric matrix. Noting that $|\tilde{\boldsymbol{a}}|^2 = m(1+\eta)$, we can write

$$D^H \Pi_c D = |\tilde{\boldsymbol{a}}|^2 [J + K] \tag{23}$$

where J and K are given by

$$J = \frac{4\pi^2}{m} \begin{bmatrix} S_{\phi\phi}\cos^2\psi & S_{\phi\psi}\cos\psi \\ S_{\phi\psi}\cos\psi & S_{\psi\psi} \end{bmatrix} \tag{24}$$

$$K = \frac{1}{1+\eta} \begin{bmatrix} \cos^2\psi & 0 \\ 0 & 1 \end{bmatrix}. \tag{25}$$

Finally, a simple rearrangement of (10) gives the CRB as

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{1}{2N}\,\frac{1}{m\rho\rho'}\left(1+\frac{1}{m\rho\rho'}\right)[J+K]^{-1} \tag{26}$$

where $\rho = \sigma_s^2/\sigma_p^2$ is the SNR at each pressure sensor and $\rho' = (1 + 1/\eta)$ is the SNR increase factor discussed in [8].

The above expression is derived under the assumption that the ratio of the two noise powers η is known. Using the unnormalized data $\boldsymbol{y}(t)$, calculation of the off-diagonal entries of the full FIM by Bangs' formula (see Appendix A) shows that those involving either of the DOA parameters and any one of σ_s^2, σ_p^2, and σ_v^2 are zero. The FIM is thus block diagonal and so $\mathrm{CRB}(\boldsymbol{\theta})$ is independent of knowledge of σ_s^2, σ_p^2, and σ_v^2. Therefore, our expression is valid whether or not η is known. Furthermore, the block diagonal structure of the FIM means that the result we have derived using the concentrated CRB of [2, Theorem 3.1] is in fact the inverse of the appropriate 2×2 block of the FIM. Note that this would not generally be the case for more than one source. We can also obtain the CRB expression for the equivalent pressure-sensor array by setting $\boldsymbol{h} = [1, 0, 0, 0]^T$, resulting in

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{1}{2N}\frac{1}{m\rho}\left(1+\frac{1}{m\rho}\right)J^{-1}. \tag{27}$$

IV. Array Design

A. Uncoupled Bound

We have already shown that the FIM is block diagonal. If the off-diagonal elements of the 2×2 block of the FIM corresponding to $\boldsymbol{\theta}$ are zero, then the azimuth and elevation are uncoupled in the bound. This means that the minimum estimation variance of one parameter is not increased by uncertainty in the other. In this case, the azimuth and elevation are the "natural" parameterization of the DOA in the sense that the information the data provides about one is independent of the information it provides about the other. We may think of them as being a canonical representation of the bearing in much the same way as we consider the eigenvectors of a linear transformation to be a canonical representation of the transformation. Furthermore, if the diagonal elements of the FIM are held constant, then the CRB of both parameters is minimized by making the off-diagonal elements zero, as illustrated in [16]. For both the vector-sensor and pressure-sensor arrays, this implies setting the off-diagonal term of the symmetric matrix J [defined in (24)] to zero. Since both diagonal and off-diagonal entries of J depend on the bearing and the array geometry, the optimum array configuration can only be found for a particular $\boldsymbol{u}$. However, from this discussion, it is clear that choosing the sensor locations such that the off-diagonal element of J is zero for all directions is a compelling source-independent design criterion.

Theorem IV.1: The following conditions are necessary and sufficient for the azimuth and elevation to be uncoupled in the CRB for all $[\phi, \psi] \in \boldsymbol{\Theta}$:

$$\sum_{j=1}^{m} r_{jx}^2 = \sum_{j=1}^{m} r_{jy}^2 \tag{28}$$

$$\sum_{j=1}^{m} r_{jx}r_{jy} = \sum_{j=1}^{m} r_{jx}r_{jz} = \sum_{j=1}^{m} r_{jy}r_{jz} = 0 \tag{29}$$

where r_{jx}, r_{jy}, and r_{jz} are the x, y, and z components of the jth sensor's position vector. If either of these conditions is not satisfied, then the azimuth and elevation are coupled on a full Lebesgue measure subset of $\boldsymbol{\Theta}$.

Proof: See Appendix B.

This theorem holds for pressure-sensor as well as vector-sensor arrays since the off-diagonal term of the CRB depends only on the off-diagonal term of the matrix J for both arrays. These are the same set of sufficient conditions given for the case of a pressure-sensor array in [16]; however, we have shown them to also be necessary and to apply to vector-sensor arrays. Arrays that exhibit this property include linear arrays, regular circular and regular square arrays, and 3-D arrays constructed from parallel combinations of such circular or square arrays (see [16]).

B. Mean-Square Angular Error

Estimating both azimuth and elevation is equivalent to estimating the bearing $\boldsymbol{u}$. A very natural measure of estimator performance is the mean square angular error, i.e., $\mathrm{E}\delta^2$ where δ is the angle between $\boldsymbol{u}$ and its estimator [17]. It is independent of the choice of reference coordinate frame and provides a single overall measure of performance. Furthermore, it does not suffer from the singularity inherent in the spherical coordinate system. It is of great interest to lower bound this quantity in order to provide a guide to achievable performance and estimator efficiency and to form an algorithm-independent design criterion. In [19], it is shown that a lower bound on the MSAE of unit-length (locally) unbiased estimators of $\boldsymbol{u}$ is

$$\mathrm{MSAE_B}(\boldsymbol{\theta}) \triangleq \cos^2\psi\cdot\mathrm{CRB}(\phi) + \mathrm{CRB}(\psi). \tag{30}$$

This bound is also independent of the choice of reference coordinate frame, however, it is not tight for a finite number of snapshots. We can extend it to a larger class of estimators, and make it tight in many cases, by considering the asymptotic normalized mean-square angular error (ANMSAE) defined (in [17]) by

$$\mathrm{ANMSAE} \triangleq \lim_{N\to\infty} N\mathrm{E}\delta^2. \tag{31}$$

In [17], it is shown that this quantity is lower bounded, for a large class of so-called regular bearing estimators, by

$$\mathrm{ANMSAE_B}(\boldsymbol{\theta}) \triangleq N(\cos^2\psi\cdot\mathrm{CRB}(\phi) + \mathrm{CRB}(\psi)). \tag{32}$$

This is a tight bound for the ANMSAE of any second-order asymptotically efficient regular estimator (e.g., the ML estimator under mild regularity conditions). Indeed, conventional beamforming and Capon's method attain the CRB asymptotically with one source [8], so the bound will be tight in these cases. Like the $\mathrm{MSAE_B}$, this bound is invariant to the choice of reference coordinate frame. Finally, note that it is not a function of N as long as the CRB is $O(1/N)$, which is the case if snapshots are independent.

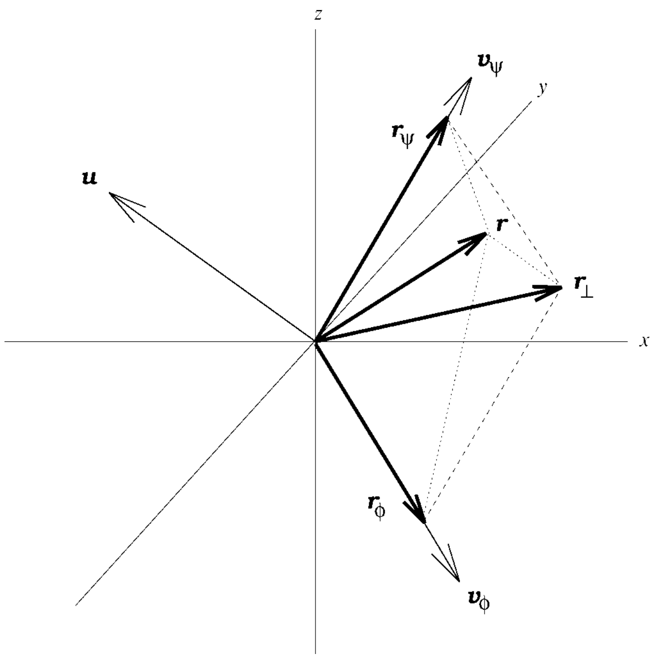

Fig. 1. Projections of a position vector onto $\boldsymbol{v}_\phi, \boldsymbol{v}_\psi$ and the subspace they span. The position vector affects the CRB and $\mathrm{ANMSAE_B}$ through the squares and products of $|\boldsymbol{r}_\phi|$, $|\boldsymbol{r}_\psi|$, and $|\boldsymbol{r}_\perp|$ [see (22)].

The dependence of the CRB, and hence the $\mathrm{ANMSAE_B}$, on array geometry is contained solely in the entries of the matrix J, specifically in $S_{\phi\phi}$, $S_{\psi\psi}$, and $S_{\phi\psi}$ defined in (22). Now $S_{\phi\phi}$ and $S_{\psi\psi}$ are the sum of squares of the lengths of the position vectors' projections onto $\boldsymbol{v}_\phi$ and $\boldsymbol{v}_\psi$, respectively, while $S_{\phi\psi}$ is the sum of products of the projections onto $\boldsymbol{v}_\phi$ and $\boldsymbol{v}_\psi$. In [8], we called $S_{\phi\phi}$ and $S_{\psi\psi}$ squared projected apertures. Since $\boldsymbol{u}$, $\boldsymbol{v}_\phi$, and $\boldsymbol{v}_\psi$ are orthonormal, we suggested that $S_{\phi\phi}$ and $S_{\psi\psi}$ could be thought of as the squared aperture "seen" by the source in two orthogonal directions. Let us also define $S_\perp \triangleq S_{\phi\phi} + S_{\psi\psi}$, which will appear in the expression for the $\mathrm{ANMSAE_B}$. To determine the geometrical interpretation of $S_\perp$, consider that it may be written

$$S_\perp = \sum_j \boldsymbol{r}_j^T[\boldsymbol{v}_\phi \boldsymbol{v}_\phi^T + \boldsymbol{v}_\psi \boldsymbol{v}_\psi^T]\boldsymbol{r}_j. \tag{33}$$

Since $\boldsymbol{u}$, $\boldsymbol{v}_\phi$, and $\boldsymbol{v}_\psi$ are orthonormal, $[\boldsymbol{v}_\phi \boldsymbol{v}_\phi^T + \boldsymbol{v}_\psi \boldsymbol{v}_\psi^T]$ is the projection matrix onto the subspace orthogonal to the DOA $\boldsymbol{u}$. Thus, $S_\perp$ is the sum of squares of the lengths of the position vectors' projections onto the plane normal to $\boldsymbol{u}$, so it may be interpreted as the total squared aperture seen by the source. Fig. 1 illustrates an individual sensor's contribution to these geometrical quantities. It shows the projections of a position vector $\boldsymbol{r}$ onto $\boldsymbol{v}_\phi$ and $\boldsymbol{v}_\psi$, denoted $\boldsymbol{r}_\phi$ and $\boldsymbol{r}_\psi$, respectively. It also shows $\boldsymbol{r}_\perp$, the projection of $\boldsymbol{r}$ onto the plane orthogonal to $\boldsymbol{u}$. It is then $|\boldsymbol{r}_\phi|^2$, $|\boldsymbol{r}_\psi|^2$, and $|\boldsymbol{r}_\perp|^2$ that contribute to $S_{\phi\phi}$, $S_{\psi\psi}$, and $S_\perp$, respectively.

With these geometrical ideas in mind, we now examine the $\mathrm{ANMSAE_B}$. Using (24)–(26), and the above definitions, some calculation shows that

$$\mathrm{ANMSAE_B}(\boldsymbol{\theta}) \propto (\alpha m S_\perp + 2m^2)/ (\alpha^2[S_{\phi\phi}S_{\psi\psi} - S_{\phi\psi}^2] + \alpha m S_\perp + m^2) \tag{34}$$

where $\alpha = 4\pi^2(1+\eta)$ and the constant of proportionality just contains terms involving m, η, and ρ. Thus, the $\mathrm{ANMSAE_B}$'s dependence on geometry is only through the four quantities defined in (22). In general, these quantities, and hence the $\mathrm{ANMSAE_B}$, depend in a complex way on the azimuth and elevation. However, if the geometrical conditions of (28) and (29) hold, substitution in (22) and simplification gives

$$\begin{aligned} S_{\phi\phi} &= \sum_j r_{jx}^2 \\ S_{\psi\psi} &= \sin^2\psi \sum_j r_{jx}^2 + \cos^2\psi \sum_j r_{jz}^2 \\ S_{\phi\psi} &= 0 \\ S_\perp &= (1+\sin^2\psi)\sum_j r_{jx}^2 + \cos^2\psi \sum_j r_{jz}^2. \end{aligned} \tag{35}$$

Consequently, $\mathrm{ANMSAE_B}(\boldsymbol{\theta})$ is independent of the azimuth and depends on the elevation only through squared trigonometric functions. One implication of this last fact is that $\mathrm{ANMSAE_B}(\boldsymbol{\theta})$ is the same for $\pm\psi$. If we extend (28) to

$$\sum_{j=1}^m r_{jx}^2 = \sum_{j=1}^m r_{jy}^2 = \sum_{j=1}^m r_{jz}^2 \tag{36}$$

while still satisfying (29), then we obtain

$$\begin{aligned} S_{\phi\phi} = S_{\psi\psi} &= \sum_j r_{jx}^2 \\ S_{\phi\psi} &= 0 \\ S_\perp &= 2\sum_j r_{jx}^2. \end{aligned} \tag{37}$$

The result is that $\mathrm{ANMSAE_B}(\boldsymbol{\theta})$ is independent of the DOA, i.e., the optimum performance is isotropic. This is a desirable characteristic in many array processing scenarios. A regular cubical array, symmetric about all three axes, is an example of a geometry that exhibits this isotropic optimum performance. Since the geometry-dependent terms of the $\mathrm{ANMSAE_B}$ come from the matrix J, which appears in the CRB expressions for both vector-sensor and pressure-sensor arrays, the sufficiency of (28) and (29) for azimuthal independence and of (36) and (29) for complete DOA independence, is equally true for both types of arrays. Finally, we note that all the results of this section apply equally to $\mathrm{MSAE_B}$ because of its similarity of structure to $\mathrm{ANMSAE_B}$.

V. Conclusion

We have shown how the optimal performance of an acoustic vector-sensor array is affected by its geometry. We derived an expression (26) for the CRB on the location parameters of a single far-field source in a homogeneous wholespace. This result was examined in great depth in [8]. We also showed that the optimal location estimation performance is independent of the knowledge of the signal and noise power parameters. We then derived a set of necessary and sufficient conditions (Theorem IV.1) that ensure the azimuth and elevation are uncoupled from each other in the bound, and it was argued that this provides a compelling array design criterion. We then derived a bound on the asymptotic mean square angular error and showed that it depends on array geometry only through the squared projections of the sensors' position vectors onto lines and planes orthogonal to the source (34). Utilizing this result, we showed that the same set of conditions (28) and (29) that ensure the location parameters are uncoupled in the CRB are also sufficient to make this bound independent of azimuth. Finally, we extended those geometrical conditions to find constraints on sensor location [see (28) and (36)] that are sufficient to ensure that the optimal performance is isotropic. While we did not show that these conditions are also necessary, it seems probable that they are; it would be interesting to determine whether that is the case. In practice, there are many factors involved in the choice of a particular arrangement of sensors, however, the conditions herein are not too restrictive as they basically impose some kind of symmetry on the structure, and, when nothing is known about the source location *a priori*, their use should give the best possible overall performance. There should therefore be strong justification for any design that deviates from these constraints.

Appendix A

Bangs' formula (see, for example, [20, p. 525]) gives an expression for the entries of the FIM for complex Gaussian distributed data. Using this formula, we shall show that the off-diagonal (coupling) entries of the FIM between the azimuth (and also elevation) and each of the three power parameters σ_s^2, σ_p^2, and σ_v^2 are zero. Let $\boldsymbol{\vartheta} = [\boldsymbol{\theta}^T, \sigma_s^2, \sigma_p^2, \sigma_v^2]^T$ be the vector of unknown parameters. Then, Bangs' formula gives the (i,j)th entry of the full 5×5 FIM as

$$[\mathrm{FIM}(\boldsymbol{\vartheta})]_{i,j} = \mathrm{tr}\left[R^{-1}\,\frac{\partial R}{\partial \vartheta_i}\,R^{-1}\,\frac{\partial R}{\partial \vartheta_j}\right] \tag{38}$$

where R is the data covariance matrix given by (7) and tr indicates the matrix trace. Using the matrix inversion lemma

$$R^{-1} = \Sigma^{-1}\left[I - \boldsymbol{a}\,\frac{\sigma_s^2}{1+\sigma_s^2\boldsymbol{a}^H\Sigma^{-1}\boldsymbol{a}}\,\boldsymbol{a}^H\Sigma^{-1}\right] \tag{39}$$

where Σ is the noise covariance matrix given by (6). Now

$$\boldsymbol{a}^H\Sigma^{-1}\boldsymbol{a} = (\boldsymbol{a}_{\mathrm{P}}\otimes\boldsymbol{h})^H I_m\otimes\begin{bmatrix}\sigma_p^2 & 0\\ 0 & I_3/\sigma_v^2\end{bmatrix}(\boldsymbol{a}_{\mathrm{P}}\otimes\boldsymbol{h}) \tag{40}$$

$$= m\rho'/\sigma_p^2 \tag{41}$$

where ρ' is defined in (26). Thus

$$R^{-1} = \Sigma^{-1}\left[I - \frac{\sigma_s^2}{1+m\rho\rho'}\,\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1}\right]. \tag{42}$$

Taking the derivatives of R, we find

$$\partial R/\partial\phi = \sigma_s^2\left[\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\right] \tag{43}$$

$$\partial R/\partial\psi = \sigma_s^2\left[\frac{\partial\boldsymbol{a}}{\partial\psi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\psi}\right] \tag{44}$$

$$\partial R/\partial\sigma_s^2 = \boldsymbol{a}\,\boldsymbol{a}^H \tag{45}$$

$$\partial R/\partial\sigma_p^2 = I_m\otimes\begin{bmatrix}1 & 0\\ 0 & 0I_3\end{bmatrix} \tag{46}$$

$$\partial R/\partial\sigma_v^2 = I_m\otimes\begin{bmatrix}0 & 0\\ 0 & I_3\end{bmatrix}. \tag{47}$$

Equations (42)–(47) give all the information necessary to calculate the entries of the FIM. We consider first the coupling between ϕ and σ_s^2. Denoting $k = \sigma_s^2/(1+m\rho\rho')$,

$$\begin{aligned}
&[\mathrm{FIM}(\boldsymbol{\vartheta})]_{\phi,\sigma_s^2}\\
&\quad= \mathrm{tr}\Big[\Sigma^{-1}(I - k\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1})\\
&\qquad\cdot\left(\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\right)\Sigma^{-1}(I - k\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1})\boldsymbol{a}\,\boldsymbol{a}^H\Big]\\
&\quad\propto \mathrm{tr}\Big[\Sigma^{-1}(I - k\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1})\\
&\qquad\cdot\left(\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\right)\Sigma^{-1}\boldsymbol{a}\,\boldsymbol{a}^H\Big]\\
&\quad= \mathrm{tr}\Big[\Sigma^{-1}\Big(k'\,\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + (1-kk')\boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\,\Sigma^{-1}\boldsymbol{a}\boldsymbol{a}^H\\
&\qquad -kk'\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1}\,\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H\Big)\Big]
\end{aligned} \tag{48}$$

where $k' = \boldsymbol{a}^H\Sigma^{-1}\boldsymbol{a} = m\rho'/\sigma_p^2$. Now

$$\begin{aligned}
&\boldsymbol{a}^H\Sigma^{-1}\,\frac{\partial\boldsymbol{a}}{\partial\phi}\\
&\quad= (\boldsymbol{a}_{\mathrm{P}}\otimes\boldsymbol{h})^H\left\{I_m\otimes\begin{bmatrix}1/\sigma_p^2 & 0\\ 0 & I_3/\sigma_s^2\end{bmatrix}\right\}\\
&\qquad\cdot\left(\frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi}\otimes\boldsymbol{h} + \boldsymbol{a}_{\mathrm{P}}\otimes\frac{\partial\boldsymbol{h}}{\partial\phi}\right)\\
&\quad= \left(\boldsymbol{a}_{\mathrm{P}}\otimes\begin{bmatrix}\frac{1}{\sigma_p^2}\\ \frac{\boldsymbol{u}}{\sigma_v^2}\end{bmatrix}\right)^H\left(\frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi}\otimes\boldsymbol{h} + \boldsymbol{a}_{\mathrm{P}}\otimes\frac{\partial\boldsymbol{h}}{\partial\phi}\right)\\
&\quad= \boldsymbol{a}_{\mathrm{P}}^H\,\frac{\partial\boldsymbol{a}_{\mathrm{P}}}{\partial\phi}\begin{bmatrix}\frac{1}{\sigma_p^2}\\ \frac{\boldsymbol{u}}{\sigma_v^2}\end{bmatrix}^H\boldsymbol{h} + \boldsymbol{a}_{\mathrm{P}}^H\boldsymbol{a}_{\mathrm{P}}\begin{bmatrix}\frac{1}{\sigma_p^2}\\ \frac{\boldsymbol{u}}{\sigma_v^2}\end{bmatrix}^H\begin{bmatrix}0\\ \boldsymbol{v}_\phi\cos\psi\end{bmatrix}\\
&\quad= 0
\end{aligned} \tag{49}$$

where the last step follows from the orthogonality of $\boldsymbol{a}_{\mathrm{P}}$ and its derivatives (16) and the orthogonality of $\boldsymbol{u}$ and $\boldsymbol{v}_\phi$. Thus, the last two terms in the third line of (48) are zero. The first term of this line is

$$k'\mathrm{tr}\left[\Sigma^{-1}\,\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H\right] = k'\boldsymbol{a}^H\Sigma^{-1}\frac{\partial\boldsymbol{a}}{\partial\phi} = 0. \tag{50}$$

Hence $[\mathrm{FIM}(\boldsymbol{\vartheta})]_{\phi,\sigma_s^2} = 0$ and so the azimuth and signal power are uncoupled in the bound.

The cross-coupling between azimuth and pressure-sensor noise power is

$$\begin{aligned}
&[\mathrm{FIM}(\boldsymbol{\vartheta})]_{\phi,\sigma_p^2}\\
&\quad= \mathrm{tr}\Big[\Sigma^{-1}(I - k\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1})\\
&\qquad\cdot\left(\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\right)\Sigma^{-1}(I - k\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1})\,\frac{\partial R}{\partial\sigma_p^2}\Big]\\
&\quad\propto \mathrm{tr}\Big[\Sigma^{-1}(I - k\boldsymbol{a}\,\boldsymbol{a}^H\Sigma^{-1})\\
&\qquad\cdot\left(\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\right)(I - k\Sigma^{-1}\boldsymbol{a}\,\boldsymbol{a}^H)\,\frac{\partial R}{\partial\sigma_p^2}\Big]\\
&\quad\propto \mathrm{tr}\Big[\Sigma^{-1}\left(\frac{\partial\boldsymbol{a}}{\partial\phi}\,\boldsymbol{a}^H + \boldsymbol{a}\,\frac{\partial\boldsymbol{a}^H}{\partial\phi}\right)\frac{\partial R}{\partial\sigma_p^2}\Big]
\end{aligned} \tag{51}$$

where the second line follows from the fact that $\Sigma^{-1}\partial R/\partial\sigma_p^2 = (\partial R/\partial\phi)/\sigma_p^2$ and the third line from using (49). Similarly to (49), it is not difficult to show that $\boldsymbol{a}(\partial R/\partial\sigma_p^2)\partial\boldsymbol{a}^H/\partial\phi$ is also zero, thus

$$[\text{FIM}(\boldsymbol{\vartheta})]_{\phi,\sigma_p^2} \propto \boldsymbol{a}^H \frac{\partial R}{\partial\phi}\frac{\partial\boldsymbol{a}}{\partial\phi} + \frac{\partial\boldsymbol{a}^H}{\partial\phi}\frac{\partial R}{\partial\phi}\boldsymbol{a} = 0. \tag{52}$$

To show that the azimuth and velocity-sensor noise power are uncoupled, we proceed in a very similar manner to the previous case, using the results $\Sigma^{-1}\partial R/\partial\sigma_v^2 = (\partial R/\partial\sigma_v^2)/\sigma_v^2$ and $\boldsymbol{a}(\partial R/\partial\sigma_v^2)\partial\boldsymbol{a}^H/\partial\phi = 0$. We have thus shown that the azimuth is uncoupled from each of the three noise parameters. The results for elevation are the same; we merely replace derivatives with respect to ϕ with derivatives with respect to ψ in the above. The result is that the 5×5 FIM has block diagonal structure

$$\text{FIM}(\boldsymbol{\vartheta}) = \begin{bmatrix}\text{FIM}(\boldsymbol{\theta}) & 0 \\ 0 & \text{FIM}([\sigma_s^2,\sigma_p^2,\sigma_v^2]^T)\end{bmatrix}. \tag{53}$$

APPENDIX B

In this appendix, we prove Theorem IV.1. From (24)–(26), the azimuth and elevation are uncoupled in the CRB are zero if and only if the off-diagonal entry of the real symmetric matrix J is zero. This term is

$$J_{\phi\psi} \propto \sum_j (\boldsymbol{r}_j^T\boldsymbol{v}_\phi)(\boldsymbol{r}_j^T\boldsymbol{v}_\psi) \tag{54}$$

$$= \left(\frac{\sin 2\phi}{2}\sum_j (r_{jx}^2 - r_{jy}^2) - \cos 2\phi\sum_j r_{jx}r_{jy}\right)\sin\psi - \left(\sin\phi\sum_j r_{jx}r_{jz} - \cos\phi\sum_j r_{jy}r_{jz}\right)\cos\psi \tag{55}$$

where the sums are over the number of sensors m and the constant of proportionality is $4\pi^2\cos\psi/m$. Sufficiency follows easily by substitution, so we consider necessity. The term $\cos\psi$ in the constant of proportionality makes $J_{\phi\psi}$ zero on the set $S_0 = [0,2\pi)\times[-\pi/2,\pi/2]$, which is a set of Lebesgue measure zero corresponding to just two directions in the spherical coordinate system. Thus, rewriting (55), we have $J_{\phi\psi} = 0\ \forall\boldsymbol{\theta}\in\boldsymbol{\Theta}$ if and only if

$$\sin\psi R_1\sin(2\phi-\alpha_1) - \cos\psi R_2\sin(\phi-\alpha_2) = 0 \tag{56}$$

for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}\cap\boldsymbol{S}_0^c$, where

$$R_1^2 = \frac{\left(\sum_j (r_{jx}^2 - r_{jy}^2)\right)^2}{4} + \left(\sum_j r_{jx}r_{jy}\right)^2 \tag{57}$$

$$R_2^2 = \left(\sum_j r_{jx}r_{jz}\right)^2 + \left(\sum_j r_{jy}r_{jz}\right)^2 \tag{58}$$

and $\alpha_1,\alpha_2\in[0,2\pi)$ with tangents

$$\tan\alpha_1 = \frac{2\left(\sum_j r_{jx}r_{jy}\right)}{\sum_j (r_{jx}^2 - r_{jy}^2)},\quad \tan\alpha_2 = \frac{\sum_j r_{jy}r_{jz}}{\sum_j r_{jx}r_{jz}}. \tag{59}$$

We may reduce (56) further to

$$R_T\sin(\psi-\alpha_T) = 0 \qquad \forall\boldsymbol{\theta}\in\boldsymbol{\Theta}\cap\boldsymbol{S}_0^c \tag{60}$$

where $\alpha_T\in[0,2\pi)$ and

$$R_T^2 = R_1^2\sin^2(2\phi-\alpha_1) + R_2^2\sin^2(\phi-\alpha_2) \tag{61}$$

$$\tan\alpha_T = \frac{R_2\sin(\phi-\alpha_2)}{R_1\sin(2\phi-\alpha_1)}. \tag{62}$$

Hence, we require $R_T = 0$ for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}\cap\boldsymbol{S}_0^c\cap\boldsymbol{S}_1^c$, where $\boldsymbol{S}_1$ is the set on which $\sin(\psi-\alpha_T) = 0$. Note that since $R_1, R_2, \alpha_1, \alpha_2$ are independent of $\boldsymbol{\theta}$, R_T and α_T are single valued functions of ϕ only. So, for each ϕ, there corresponds exactly one $\psi\in[-\pi/2,\pi/2]$, such that $[\phi,\psi]\in S_1$ (unless for some $\phi=\phi'$ say, $\alpha_T = 3\pi/2$, in which case $\{\phi'\}\times\{-\pi/2,\pi/2\}\in S_1$). Hence, S_1 is a set of Lebesgue measure zero.

Now there are exactly four ways that R_T can be zero:

1) $\sin(2\phi-\alpha_1) = 0$ and $\sin(\phi-\alpha_2) = 0$;
2) $R_1 = 0$ and $\sin(\phi-\alpha_2) = 0$;
3) $R_2 = 0$ and $\sin(2\phi-\alpha_1) = 0$;
4) $R_1 = 0$ and $R_2 = 0$

Since α_1,α_2 are constants, sets S_2 and S_3 corresponding to the zeros of the sine terms in 2) and 3) are lines of measure zero in $\boldsymbol{\Theta}$, while the set corresponding to 1) is $S_2\cap\boldsymbol{S}_3$. Hence, we require 4) to hold on the full measure set $=\boldsymbol{\Theta}\cap\boldsymbol{S}_0^c\cap S_1^c\cap\boldsymbol{S}_2^c\cap S_3^c$. From (57) and (58), we see that 4) is satisfied only if the stated conditions (28) and (29) hold. Note that the off-diagonal entry of J will be nonzero for all $\boldsymbol{\Theta}$, except on the zero measure subset $S_0\cup\boldsymbol{S}_1\cup S_2\cup S_3$, if either one of (28) or (29) fails to hold. ■

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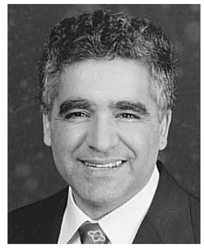

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