array segment angle estimates shows that the best (in the sense of least mean squared error) estimator performance is obtained when

- The array shape is linear and the source direction is broadside.
- The wavelength of the source is small.
- The number of snapshots is large and the SNR is high.

Note that wildly fluctuating array shapes suffer from position ambiguity problems as well as a high CRLB. Also phase wrapping ambiguity could limit the minimum source wavelength in a practical application.

The CRLB is relatively independent of the number of receivers used to generate the correlation matrix. This suggests that an estimator with comparable performance but greatly reduced computational cost could be designed around an array partitioned into small groups of receivers. The paper also shows that the CRLB on the array segment angles is significantly greater if the source direction is unknown compared to a known source direction.

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Uniqueness Study of Measurements Obtainable with Arrays of Electromagnetic Vector Sensors

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***Abstract*—In this correspondence, we investigate linear dependence of steering vectors for arrays comprising multiple electromagnetic vector sensors. We derive upper and lower bounds for the number of linearly independent steering vectors associated with such arrays. These bounds are potentially useful for determining the number of signals whose directions-of-arrival can be uniquely identified.**

I. Introduction

There have been a few studies on direction-of-arrival (DOA) estimation with electromagnetic (EM) vector sensors [1]–[5]. One practically important question concerning this subject is the number of signals whose DOA's can be uniquely determined. To address this question, one needs a good characterization of linear dependence of steering vectors [6]–[9]. Tan *et. al.* [5] have established that, for a vector sensor, every three steering vectors with distinct DOA's are linearly independent. However, equivalent results for the case of multiple vector sensors have yet to be found. In this correspondence, we shall present some findings on linear dependence of steering vectors for arrays with multiple vector sensors (which we shall refer to as vector-sensor arrays).

The findings obtained here can be used with the theories on source identifiability developed in [8] and [9] to determine useful bounds for the number of EM signals whose DOA's can be uniquely determined. Note that the theories of [8] are applicable to general multiple parameters and signals per source model (see [8] for elaboration), while those of [9] are only for signals measured with EM vector sensors.

II. Data Model

The steering vector of an array comprising m vector sensors, corresponding to a polarized signal with azimuth ϕ_k, elevation ψ_k, orientation angle α_k, and ellipticity angle β_k, can be expressed [1], [2] as follows:

$$\mathbf{a}(\boldsymbol{\theta_k}) = \mathbf{d}(\phi_k, \psi_k) \otimes \mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k) \tag{1}$$

where $\boldsymbol{\theta_k} = (\phi_k, \psi_k, \alpha_k, \beta_k)$,

$$\mathbf{B}(\phi_k, \psi_k) = \begin{pmatrix} -\sin\phi_k & -\cos\phi_k \sin\psi_k \\ \cos\phi_k & -\sin\phi_k \sin\psi_k \\ 0 & \cos\psi_k \\ -\cos\phi_k \sin\psi_k & \sin\phi_k \\ -\sin\phi_k \sin\psi_k & -\cos\phi_k \\ \cos\psi_k & 0 \end{pmatrix} \tag{2}$$

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$$\mathbf{Q}(\alpha_k) = \begin{pmatrix} \cos\alpha_k & \sin\alpha_k \\ -\sin\alpha_k & \cos\alpha_k \end{pmatrix} \quad (3)$$

$$\mathbf{w}(\beta_k) = [\cos\beta_k, j\sin\beta_k]^T \quad (4)$$

$$\mathbf{d}(\phi_k, \psi_k) = [e^{-j2\pi\mathbf{r}_1\cdot\mathbf{u}(\phi_k,\psi_k)/\lambda}, \ldots, e^{-j2\pi\mathbf{r}_m\cdot\mathbf{u}(\phi_k,\psi_k)/\lambda}]^T \quad (5)$$

$$\mathbf{u}(\phi_k, \psi_k) = \begin{pmatrix} \cos\phi_k\cos\psi_k \\ \sin\phi_k\cos\psi_k \\ \sin\psi_k \end{pmatrix} \quad (6)$$

$\mathbf{r}_l$ is the position vector of the lth sensor, λ the wavelength of the signal, "·" the dot product and $\otimes$ the Kronecker product. Note that $\mathbf{d}(\phi, \psi)$ is the steering vector, corresponding to DOA (ϕ, ψ), of a scalar-sensor array that has the same sensor configuration as the vector-sensor array.

III. A Lower Bound for the Number of Linearly Independent Steering Vectors

We shall first establish a result that relates linear independence of steering vectors of vector-sensor arrays to that of scalar-sensor arrays having the same sensor configuration. This is useful in the sense that it provides a means of extending findings on the linear dependence of steering vectors for scalar-sensor arrays [10]–[13] to vector-sensor arrays. We shall first state a useful result.

Theorem 1—Tan et. al. [5]: For any vector-sensor array, every three steering vectors with distinct DOA's are linearly independent. In addition, four steering vectors of a vector-sensor array corresponding to distinct DOA's are linearly *dependent* only if the ellipticity angles of the signals corresponding to the steering vectors are identical.

We are now ready to establish a new theorem.

Theorem 2: Consider $k \geq 4$ distinct DOA's $(\phi_1, \psi_1), \ldots, (\phi_k, \psi_k)$. Then the steering vectors $\mathbf{a}(\boldsymbol{\theta}_1), \ldots, \mathbf{a}(\boldsymbol{\theta}_k)$ of a vector-sensor array, where $\mathbf{a}(\boldsymbol{\theta})$ is defined in (1), are linearly independent if

$$rank\,[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_k, \psi_k)] \geq k - 2$$

where $\mathbf{d}(\phi, \psi)$ is defined in (5). Moreover, given that not more than three of the ellipticity angles $\beta_1, \ldots, \beta_k$ are identical, then the steering vectors $\mathbf{a}(\boldsymbol{\theta}_1), \ldots, \mathbf{a}(\boldsymbol{\theta}_k)$ are linearly independent if

$$rank\,[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_k, \psi_k)] \geq k - 3.$$

Proof: See Appendix A.

Corollary: Consider a scalar-sensor array and a vector-sensor array that have the same sensor configuration, and suppose that every k steering vectors of the scalar-sensor array that correspond to distinct DOA's are linearly independent, i.e.

$$rank\,[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_k, \psi_k)] = k \quad (7)$$

for all distinct DOA's $(\phi_1, \psi_1), \ldots, (\phi_k, \psi_k)$. Then every $(k + 2)$ steering vectors of the vector-sensor array that correspond to distinct DOA's are linearly independent. Moreover, every $(k + 3)$ steering vectors of the vector-sensor array that correspond to distinct DOA's are linearly independent if not more than three of the $(k+3)$ steering vectors have identical ellipticity angle.

Remarks: This theorem establishes that the steering vectors of a vector-sensor array have a higher order of linear independence than those of a scalar-sensor array with the same sensor configuration.

It also provides a lower bound on the number of linearly independent steering vectors. Indeed, if every k steering vectors of the scalar-sensor array with distinct DOA's are linearly independent, then every $(k+2)$ steering vectors of the vector-sensor array are linearly independent. For $(k + 3)$ or more steering vectors, it is unknown whether they are linearly independent without *a priori* knowledge about the ellipticity angles of the signals associated with the steering vectors. In any case, characterization of linear independence of $(k+4)$ or more steering vectors remains unresolved.

Proof of the Corollary: See Appendix B.

The converse of the above corollary is not true. Indeed, consider a vector-sensor array and a scalar-sensor array, both of which have the same sensor configuration. Then the fact that every $(k + 2)$ steering vectors of the vector-sensor array are linearly independent does not necessarily imply that every k steering vectors of the scalar-sensor array are linearly independent. To see this, consider an array comprising three noncollinear scalar sensors with intersensor spacings all less than $\lambda/2$. Then there exist two, but not more than two, distinct DOA's that give rise to the same steering vector (see [10]). Consequently, we have

$$rank\,[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_4, \psi_4)] \geq 2$$

for all distinct DOA's $(\phi_1, \psi_1), \ldots, (\phi_4, \psi_4)$. Thus, it follows immediately from Theorem 2 that for the vector-sensor array with the same sensor configuration, every four steering vectors with distinct DOA's are linearly independent. However, as far as the scalar-sensor array is concerned, there exist two linearly dependent steering vectors with distinct DOA's.

IV. Upper Bounds for the Number of Linearly Independent Steering Vectors

In this section, we derive upper bounds for the number of linearly independent steering vectors of a vector-sensor array.

Theorem 3: Consider an array of vector sensors. If there exist $(3k + 1)$ distinct DOA's $(\phi_1, \psi_1), \ldots, (\phi_{3k+1}, \psi_{3k+1})$ such that

$$rank\,[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_{3k+1}, \psi_{3k+1})] \leq k \quad (8)$$

then there exist $(3k + 1)$ steering vectors with distinct DOA's that are linearly dependent.

Remarks: This theorem provides an upper bound for the number of linearly independent steering vectors. Indeed, given a scalar-sensor array with steering vectors satisfying (8), then at most every $3k$ steering vectors of a vector-sensor array with the same sensor configuration are linearly independent.

The upper bound and the lower bound given in the corollary to Theorem 2 are derived based on some assumed conditions specified by (8) and (7), respectively, concerning linear independence of steering vectors of the scalar-sensor array that has the same sensor configuration as the vector-sensor array of interest. Whether conditions (7) and (8) can be satisfied depends on not only the number of sensors, but also the actual sensor configurations. Since arrays with general sensor configurations are considered here, the bounds are not specified in terms of the number of sensors.

Proof of Theorem 3: See Appendix C.

We can apply Theorem 3 to obtain another result.

Theorem 4: Consider an array comprising $m\ (\geq 2)$ vector sensors, of which k lie on a straight line. Then there exist $3(m - k + 1) + 1$ steering vectors with distinct DOA's that are linearly dependent.

Remark: It has been established in [5] that for an array comprising $m \geq 2$ vector sensors there exist $(3m+1)$ steering vectors with distinct DOA's that are linearly dependent. By Theorem 4 and on the basis that such an array contains at least two collinear sensors, we deduce that there exist $(3m-2)$ linearly dependent steering vectors that correspond to distinct DOA's. Since the number $(3m-2)$ is less than $(3m+1)$ by three, Theorem 4 offers a slightly tighter upper bound on the number of linearly independent steering vectors than the result of [5].

Proof of Theorem 4: See Appendix D.

Finally, we generalize Theorem 5 of [5].

Theorem 5: Given any $(6m-1)$ steering vectors of an array of m vector sensors, then for *any* DOA there exists a steering vector that is a linear combination of the $(6m-1)$ steering vectors.

Remarks: This theorem has been established in [5] for the special case of $m=1$.

It follows from this theorem that in the presence of $(6m-1)$ polarized signals, it is possible to find a steering vector corresponding to an arbitrary direction that intersects the signal subspace. Consequently, estimating the DOA's of $(6m-1)$ polarized signals is impossible.

Proof of Theorem 5: See Appendix E.

V. Conclusion

We showed explicitly that the steering vectors of a vector-sensor array have a higher order of linear independence than those of a scalar-sensor array with the same sensor configuration. Thus, with vector-sensor arrays, there is a greater number of signals whose DOA's can be uniquely determined. We also established some upper bounds for the number of linearly independent steering vectors. These bounds are potentially useful for determining the maximum number of signals whose DOA's can be uniquely identified.

Appendix A
Proof of Theorem 2

For the first part of the theorem, we need to show that if $\mathbf{a}(\boldsymbol{\theta_1}), \ldots, \mathbf{a}(\boldsymbol{\theta_k})$ are linearly dependent where $(\phi_1, \psi_1), \ldots, (\phi_k, \psi_k)$ are distinct DOA's, then

$$rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_k,\psi_k)] < k-2.$$

Now consider an array of m vector sensors and let $\mathbf{a}(\boldsymbol{\theta_1}), \ldots, \mathbf{a}(\boldsymbol{\theta_k})$ be linearly dependent steering vectors, where $(\phi_1, \psi_1), \ldots, (\phi_k, \psi_k)$ are distinct DOA's. Then there exist $n \leq k$ linearly dependent steering vectors among k of them, say $\mathbf{a}(\boldsymbol{\theta_1}), \ldots, \mathbf{a}(\boldsymbol{\theta_n})$, such that every $(n-1)$ of them are linearly independent (The first part of Theorem 1 guarantees that $n \geq 4$). Thus, we can write

$$\sum_{l=1}^{n} c_l \mathbf{a}(\boldsymbol{\theta_l}) = \mathbf{0}$$

where $c_l \in \mathbb{C}\backslash\{0\}$, for $l = 1,\ldots,n$. Observe that rows $(6(h-1)+1)$th to $(6h)$th of the above equation, for $h=1,\ldots,m$, can be written as

$$\sum_{l=1}^{n} c_l d_{h,l} \mathbf{B}(\phi_l,\psi_l)\mathbf{Q}(\alpha_l)\mathbf{w}(\beta_l) = \mathbf{0}$$

where $d_{h,l} = e^{-j2\pi \mathbf{r}_h \cdot \mathbf{u}(\phi_l,\psi_l)}/\lambda$, and $\mathbf{B}(\phi,\psi)$, $\mathbf{Q}(\alpha)$ and $\mathbf{w}(\beta)$ are defined in (2), (3), and (4), respectively. This implies that the vectors $[c_1 d_{h,1},\ldots,c_n d_{h,n}]^T$ for $h=1,\ldots,m$ are in the null space of the matrix $\hat{\mathbf{A}}$, where

$$\hat{\mathbf{A}} = [\mathbf{B}(\phi_1,\psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1),\ldots,\mathbf{B}(\phi_n,\psi_n)\mathbf{Q}(\alpha_n)\mathbf{w}(\beta_n)]. \quad (9)$$

Since $n \geq 4$, by the first part of Theorem 1 we have $rank\,(\hat{\mathbf{A}}) > 2$. Consequently,

$$null\,(\hat{\mathbf{A}}) = col\,(\hat{\mathbf{A}}) - rank\,(\hat{\mathbf{A}}) < n-2$$

where $col\,(\hat{\mathbf{A}})$ denotes the number of columns of $\hat{\mathbf{A}}$. This implies that $rank\,\mathbf{\Gamma} < (n-2)$, where

$$\mathbf{\Gamma} = \begin{pmatrix} c_1 d_{1,1} & c_2 d_{1,2} & \cdots & c_n d_{1,n} \\ c_1 d_{2,1} & c_2 d_{2,2} & \cdots & c_n d_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 d_{m,1} & c_2 d_{m,2} & \cdots & c_n d_{m,n} \end{pmatrix} = [c_1\mathbf{d}(\phi_1,\psi_1),\ldots,c_n\mathbf{d}(\phi_n,\psi_n)].$$

Since $c_l \neq 0$, for $l=1,\ldots,n$

$$rank\,\mathbf{\Gamma} = rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_n,\psi_n)] < n-2.$$

Clearly, adding $(k-n)$ columns, namely $\mathbf{d}(\phi_{n+1},\psi_{n+1}), \ldots, \mathbf{d}(\phi_k,\psi_k)$ to the matrix $[\mathbf{d}(\phi_1,\psi_1), \ldots, \mathbf{d}(\phi_n,\psi_n)]$ would increase its rank by at most $(k-n)$. Thus, we have

$$\begin{aligned} &rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_n,\psi_n),\mathbf{d}(\phi_{n+1},\psi_{n+1}),\ldots,\mathbf{d}(\phi_k,\psi_k)] \\ &< (n-2)+(k-n) = k-2. \end{aligned}$$

This completes the proof of the first part of the theorem.

For the second part of the theorem, we need to show that if $\mathbf{a}(\boldsymbol{\theta_1}), \ldots, \mathbf{a}(\boldsymbol{\theta_k})$ are linearly dependent where $(\phi_1,\psi_1), \ldots, (\phi_k,\psi_k)$ are k distinct DOA's, and not more than three of the ellipticity angles $\beta_1, \ldots, \beta_k$ are identical, then

$$rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_k,\psi_k)] < k-3.$$

Since not more than three of the ellipticity angles $\beta_1, \ldots, \beta_k$ are identical, it follows from the second part of Theorem 1 that $rank\,(\hat{\mathbf{A}}) > 3$, where $\hat{\mathbf{A}}$ is as defined in (9). Adopting similar arguments to those in the first part of this proof, we can easily show that

$$rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_k,\psi_k)] < k-3. \quad \blacksquare$$

Appendix B
Proof of the Corollary to Theorem 2

If $k=1$, the theorem follows immediately from Theorem 1. Thus, we shall assume that $k \geq 2$. Since every k steering vectors of the scalar-sensor array that correspond to distinct DOA's are linearly independent, we have

$$rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_k,\psi_k)] = k$$

for distinct DOA's $(\phi_1,\psi_1), \ldots, (\phi_k,\psi_k)$. Thus, for distinct DOA's $(\phi_1,\psi_1), \ldots, (\phi_{k+3},\psi_{k+3})$, we have

$$\begin{aligned} &rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_{k+3},\psi_{k+3})] \\ &\geq rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_{k+2},\psi_{k+2})] \\ &\geq rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_k,\psi_k)] = k. \end{aligned}$$

Consequently, both parts of the theorem follow directly from Theorem 2. ■

Appendix C
Proof of Theorem 3

Let $rank\,[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_{3k+1},\psi_{3k+1})] = n \leq k$. We may assume, without loss of generality, that the first n columns of the matrix $[\mathbf{d}(\phi_1,\psi_1),\ldots,\mathbf{d}(\phi_{3k+1},\psi_{3k+1})]$ are linearly independent. Thus, the vectors $\mathbf{d}(\phi_l,\psi_l)$, for $l=1,\ldots,(3k+1)$ can be expressed as a linear combination of the vectors $\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_n, \psi_n)$. Hence, for $l = 1, \ldots, (3k+1)$ we can write

$$\mathbf{d}(\phi_l, \psi_l) = \mu_{l,1}\mathbf{d}(\phi_1, \psi_1) + \ldots + \mu_{l,n}\mathbf{d}(\phi_n, \psi_n) \tag{10}$$

where $[\mu_{l,1}, \ldots, \mu_{l,n}] \in \mathbb{C}^{1\times n}\backslash\{\mathbf{0}\}$. Exploiting the fact that every four steering vectors of one vector sensor corresponding to circularly polarized signals having the same spin direction are linearly dependent (see Theorem 6 of [5]), for $l = 1, \ldots, (3k+1)$ we can write

$$\begin{aligned}&\mathbf{B}(\phi_l, \psi_l)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ) = \nu_{l,1}\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ)\\ &+ \nu_{l,2}\mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ) +\\ &\nu_{l,3}\mathbf{B}(\phi_3, \psi_3)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ)\end{aligned} \tag{11}$$

where $\nu_{l,1}, \nu_{l,2}, \nu_{l,3} \in \mathbb{C}$. Furthermore, by the first part of Theorem 1, we have $\nu_{l,1}, \nu_{l,2}, \nu_{l,3} \neq 0$. Thus, taking the Kronecker product of (10) and (11), we obtain for $l = 1, \ldots, (3k+1)$

$$\begin{aligned}&\mathbf{a}(\phi_l, \psi_l, 0^\circ, 45^\circ) = \mathbf{d}(\phi_l, \psi_l) \otimes \mathbf{B}(\phi_l, \psi_l)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ)\\ &= \Big(\mu_{l,1}\mathbf{d}(\phi_1, \psi_1) + \ldots + \mu_{l,n}\mathbf{d}(\phi_n, \psi_n)\Big)\\ &\qquad \otimes \Big(\nu_{l,1}\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ)\\ &\quad + \nu_{l,2}\mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ) + \nu_{l,3}\mathbf{B}(\phi_3, \psi_3)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ)\Big)\\ &= \sum_{p=1}^{3}\sum_{i=1}^{n} \mu_{l,i}\nu_{l,p}\mathbf{d}(\phi_i, \psi_i) \otimes \mathbf{B}(\phi_p, \psi_p)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ).\end{aligned}$$

Consequently, the vectors $\mathbf{a}(\phi_1, \psi_1, 0^\circ, 45^\circ), \ldots, \mathbf{a}(\phi_{3k+1}, \psi_{3k+1}0^\circ, 45^\circ)$, are spanned by the $3n$ vectors $\mathbf{d}(\phi_i, \psi_i) \otimes \mathbf{B}(\phi_p, \psi_p)\mathbf{Q}(0^\circ)\mathbf{w}(45^\circ)$, for $i = 1, \ldots, n$, and $p = 1, \ldots, 3$. Since $3n \leq 3k < (3k+1)$, the $(3k+1)$ vectors $\mathbf{a}(\phi_1, \psi_1, 0^\circ, 45^\circ), \ldots, \mathbf{a}(\phi_{3k+1}, \psi_{3k+1}, 0^\circ, 45^\circ)$ must be linearly dependent. ■

Appendix D
Proof of Theorem 4

Without loss of generality, we may assume that the first k vector sensors are collinear. Let $(\phi_1, \psi_1), \ldots, (\phi_n, \psi_n)$ be distinct DOA's such that the vectors $\mathbf{u}(\phi_1, \psi_1), \ldots, \mathbf{u}(\phi_n, \psi_n)$, where $\mathbf{u}(\phi, \psi)$ is defined in (6) and $n = 3(m-k+1)+1$, lie on a cone with the line joining the first k collinear sensors as axis. Then it can be shown easily that $d_{h,l} = \gamma_h d_{k,l}$, for $1 \leq h \leq (k-1)$, and $1 \leq l \leq n$, where $d_{h,l} = e^{-j2\pi\mathbf{r}_h \cdot \mathbf{u}(\phi_l, \psi_l)/\lambda}$ and $\gamma_h \in \mathbb{C}\backslash\{0\}$, i.e., the 1st, ..., $(k-1)$th rows of $\Big[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_n, \psi_n)\Big]$ are scalar multiples of the kth row. As a result, the rank of $\Big[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_n, \psi_n)\Big]$ remains the same with the removal of its first $(k-1)$ rows, i.e.,

$$\begin{aligned}&rank\Big[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_n, \psi_n)\Big]\\ &\qquad = rank\Big[\hat{\mathbf{d}}(\phi_1, \psi_1), \ldots, \hat{\mathbf{d}}(\phi_n, \psi_n)\Big]\end{aligned}$$

where $\hat{\mathbf{d}}(\phi_l, \psi_l) = [d_{k,l}, \ldots, d_{m,l}]^T$. Since the rank of a matrix is less than or equal to the number of its rows, we have

$$\begin{aligned}&rank\Big[\mathbf{d}(\phi_1, \psi_1), \ldots, \mathbf{d}(\phi_n, \psi_n)\Big]\\ &= rank\Big[\hat{\mathbf{d}}(\phi_1, \psi_1), \ldots, \hat{\mathbf{d}}(\phi_n, \psi_n)\Big] \leq (m-k+1).\end{aligned}$$

By Theorem 3, there exist $3(m-k+1)+1$ steering vectors with distinct DOA's that are linearly dependent. ■

Appendix E
Proof of Theorem 5

Consider $(6m-1)$ steering vectors $\mathbf{a}(\boldsymbol{\theta_1}), \ldots, \mathbf{a}(\boldsymbol{\theta_{6m-1}})$ and a DOA (ϕ_{6m}, ψ_{6m}). Let $\mathbf{d}(\phi_{6m}, \psi_{6m}) \otimes \mathbf{B}(\phi_{6m}, \psi_{6m}) = (\mathbf{p}, \mathbf{q})$ where $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{6m\times 1}$. Then since the vectors have only $6m$ rows, we can write

$$\begin{aligned}c_1\mathbf{a}(\boldsymbol{\theta_1}) + \ldots + c_{6m-1}\mathbf{a}(\boldsymbol{\theta_{6m-1}}) &= c_{6m}\mathbf{p} + c_{6m+1}\mathbf{q}\\ &= \mathbf{d}(\phi_{6m}, \psi_{6m}) \otimes \mathbf{B}(\phi_{6m}, \psi_{6m})\begin{pmatrix} c_{6m}\\ c_{6m+1}\end{pmatrix}\end{aligned}$$

where $(c_1, \ldots, c_{6m+1}) \in \mathbb{C}^{(6m+1)\times 1}\backslash\{\mathbf{0}\}$. We can write

$$(c_{6m}, c_{6m+1})^T = \|(c_{6m}, c_{6m+1})^T\|e^{j\gamma}\mathbf{Q}(\alpha_{6m})\mathbf{w}(\beta_{6m})$$

for some $\gamma \in (-\pi, \pi]$, $\alpha_{6m} \in (-\pi/2, \pi/2]$ and $\beta_{6m} \in [-\pi/4, \pi/4]$ (see Lemma 2.1 of [1]). Thus, it follows that

$$\begin{aligned}&c_1\mathbf{a}(\boldsymbol{\theta_1}) + \ldots + c_{6m-1}\mathbf{a}(\boldsymbol{\theta_{6m-1}})\\ &= \|(c_{6m}, c_{6m+1})^T\|e^{j\gamma}\mathbf{d}(\phi_{6m}, \psi_{6m}) \otimes\\ &\qquad \mathbf{B}(\phi_{6m}, \psi_{6m})\mathbf{Q}(\alpha_{6m})\mathbf{w}(\beta_{6m})\\ &= \|(c_{6m}, c_{6m+1})^T\|e^{j\gamma}\mathbf{a}(\boldsymbol{\theta_{6m}}).\end{aligned}$$ ■

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