

uniquely determine a band-limited signal, we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |f(x_n, y_m)|^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) F_s^*(u, v) du dv \\ &= \frac{1}{T_1 T_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) F_p^*(u, v) du dv \\ &= \frac{1}{T_1 T_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) f_p^*(x, y) dx dy. \end{aligned} \quad (12)$$

IV. CONCLUSION

In this correspondence, we extended the Parseval relationship to a general nonuniform sampling set for both 1-D and 2-D signals. The Parseval relationship for uniform samples (as well as the recurring nonuniform sampling set derived for 1-D signals by [4]) happen to be special cases of our general Parseval theorem.

APPENDIX A

We would like to prove the existence of the Fourier transform of a set of nonuniform samples. If a set of nonuniform samples  $t_n$  is a sampling set, then from [5], [6], and [8] we have the following inequality:

$$\sum_{n=-\infty}^{\infty} |x(t_n)|^2 \leq AE \quad (13)$$

where  $A$  is a constant and  $E$  is the energy of the  $x(t)$ . Considering the class of band-limited  $L^2$  signals that have energies smaller than or equal to  $E$ , we have

$$\sum_{n=-\infty}^{\infty} |x(t_n)|^2 \leq C \quad (14)$$

where  $C$  is a constant. If (14) is true, then according to the Ritz-Fisher theorem [7], [8], the following series exists:

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(t_n) e^{-j2\pi f t_n}$$

$X_s(f)$  in the above is an almost periodic function in the frequency domain.  $X_s(f)$  is the same as the Fourier transform of the nonuniform samples, and hence we have proved the existence of  $X_s(f)$ .

APPENDIX B

We would like to prove the existence of the 2-D Fourier transform of a set of the nonuniform samples. If a set of the nonuniform samples  $\{x_n, y_m\}$  is a sampling set, then from the Nikol'skii inequality [9] we get

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f(x_n, y_m)|^2 \leq AE \quad (15)$$

where  $A$  is a constant and  $E$  is the energy of the signal  $f(x, y)$ . Considering the class of band-limited  $L^2(R^2)$  signals that have energies smaller than or equal to  $E$ , we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f(x_n, y_m)|^2 \leq C \quad (16)$$

where  $C$  is a constant. If (16) is true, then according to the generalized Ritz-Fisher theorem [7], the following series exists:

$$F_s(u, v) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(x_n, y_m) e^{-j2\pi u x_n} e^{-j2\pi v y_m}$$

$F_s(u, v)$  in the above is an almost periodic function in the frequency domain.  $F_s(u, v)$  is the same as the Fourier transform of the nonuniform samples, and hence we have proved the existence of  $F_s(u, v)$ .

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Frequency Domain Cramer-Rao Bound for Gaussian Processes

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**Abstract**—An expression is derived for the asymptotic frequency domain Cramer-Rao bound (CRB) on the parameters of Gaussian processes with information in their means. The general result extends Whittle's formula, which is applicable to Gaussian processes with information in their covariance matrices or spectra. It is useful in many fields, such as system identification, radar, and sonar. Examples of its application to system deconvolution and time delay estimation in colored noise are demonstrated.

I. INTRODUCTION

A general discrete time process  $y(t)$  can be decomposed into

$$y(t) = m(t) + v(t) \quad t = 1, \dots, N \quad (1)$$

where  $m(t)$  and  $v(t)$  are, respectively, the deterministic and zero-mean random components of  $y(t)$ , and  $N$  denotes the number of data samples. In parameter estimation problems, it is often assumed that  $m(t)$  and the covariance or spectrum of  $v(t)$  are functions of an unknown  $p$ -dimensional parameter vector  $\theta$ .

A useful tool for evaluating the performance of estimation techniques is the Cramer-Rao bound (CRB) on the covariance of the parameter estimate errors (see, e.g., [2]-[4]). For any unbiased estimate of  $\theta$ , the CRB is given by

$$\text{cov}(\hat{\theta}) \geq J^{-1}(\theta) \quad (2)$$

where  $J(\theta)$  denotes the Fisher information matrix (FIM) whose

Manuscript received December 5, 1988; revised September 2, 1989. The work of A. Nehorai was supported by the Air Force Office of Scientific Research under Grant AFOSR-90-0164.

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IEEE Log Number 9034991.

entries are given by

$$[J(\theta)]_{kl} = E \left[ \frac{\partial \ln p(y, \theta)}{\partial \theta_k} \frac{\partial \ln p(y, \theta)}{\partial \theta_l} \right] \quad k, l = 1, \dots, p \quad (2a)$$

and  $p(y, \theta)$  is the probability density function of  $y(t)$ .

For Gaussian  $y(t)$ , the entries of the time domain FIM can be written as

$$[J(\theta)]_{kl} = \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right]^T R^{-1}(\theta) \left[ \frac{\partial m(\theta)}{\partial \theta_l} \right] + \frac{1}{2} \text{tr} \left\{ R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_k} R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_l} \right\} \quad (3)$$

where  $\text{tr} \{ \}$  denotes trace of a matrix,  $R(\theta)$  is the  $(N \times N)$  covariance matrix of  $v(t)$ , and

$$m(\theta) = [m(1, \theta) \dots m(N, \theta)]^T \quad (N \times 1).$$

Special cases of (3) appeared in [5], [6] ( $m(t, \theta)$  independent of  $\theta$ ) and [7], [8] ( $R(\theta)$  independent of  $\theta$ ). See also [9] for an alternative proof of (3).

*Remarks:* An interesting implication of (3) is that whenever the signal  $m(t)$  and noise  $v(t)$  are independently parametrized, i.e., whenever

$$\theta = \begin{bmatrix} \rho \\ \eta \end{bmatrix} \quad m(\theta) = m(\rho) \quad R(\theta) = R(\eta) \quad (4)$$

the FIM will have a block diagonal form as follows:

$$J(\theta) = \begin{bmatrix} J_m(\rho) & 0 \\ 0 & J_r(\eta) \end{bmatrix} \quad (5)$$

where  $\rho$  and  $\eta$  are the signal and noise vector components, respectively, of  $\theta$ . The matrices  $J_m$  and  $J_r$  are the corresponding FIM's. Note that such cases are very common (e.g., sine waves in colored noise, for which an explicit asymptotic expression for the FIM was found in [10]).

Equation (3) holds for nonstationary  $v(t)$  as long as the nonstationarity can be summarized by a finite parameter set. In the following sections, it will be assumed that  $v(t)$  is a zero-mean stationary random process.

## II. THE ASYMPTOTIC FREQUENCY DOMAIN FIM

In many applications, it is useful to express the FIM in the frequency domain. Typical applications include: a) signal parameter estimation in colored noise (see an example later); b) system identification (see, e.g., [11] and [12]); and c) multipath parameter estimation (see, e.g., [13]).

Whittle's formula [1] provides the asymptotic (large  $N$ ) frequency domain expression of the second term of (3):

$$\begin{aligned} & \text{tr} \left\{ R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_k} R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_l} \right\} \\ &= \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{\frac{\partial \phi(\omega, \theta)}{\partial \theta_k} \frac{\partial \phi(\omega, \theta)}{\partial \theta_l}}{\phi^2(\omega, \theta)} d\omega \end{aligned} \quad (6)$$

where  $\phi(\omega, \theta)$  denotes the power spectral density of  $v(t)$ :

$$\phi(\omega, \theta) = \sum_{\tau=-\infty}^{\infty} r(\tau, \theta) e^{-i\omega\tau} \quad (6a)$$

where  $r(\tau, \theta)$  is the correlation function of  $v(t)$ . For completeness, a simple proof of (6) is given in Appendix A.

The purpose of the following discussion is to derive the asymptotic frequency domain FIM expression for the deterministic (mean) part of (3).

Let  $W$  be the  $(N \times N)$  discrete Fourier transform (DFT) matrix, i.e.,

$$W_{nm} = \frac{1}{\sqrt{N}} e^{im\omega_n} \quad \omega_n = \frac{2\pi}{N} n \quad m, n = 1, \dots, N. \quad (7)$$

The matrix  $W$  is unitary, thus,  $W^*W = I$ . Using this fact, we can write

$$\begin{aligned} & \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right]^T R^{-1}(\theta) \left[ \frac{\partial m(\theta)}{\partial \theta_l} \right] \\ &= \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right]^T W [W^*R(\theta)W]^{-1} W^* \left[ \frac{\partial m(\theta)}{\partial \theta_l} \right]. \end{aligned} \quad (8)$$

For  $N$  large compared to the correlation length of  $v(t)$ , the correlation between the Fourier coefficients of  $v(t)$  at different frequencies varies as  $N^{-1}$  (see, e.g., [14]) and hence is negligible. Therefore, for large  $N$

$$W^*R(\theta)W = \text{diag} \{ \phi(\omega_n, \theta) \} \quad (9)$$

i.e.,  $W^*R(\theta)W$  is asymptotically a diagonal matrix whose  $n$ th entry is  $\phi(\omega_n, \theta)$ . Thus, for large  $N$

$$\begin{aligned} & \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right]^T R^{-1}(\theta) \left[ \frac{\partial m(\theta)}{\partial \theta_l} \right] \\ &= \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right]^T W \left[ \text{diag} \left\{ \frac{1}{\phi(\omega_n, \theta)} \right\} \right] W^* \left[ \frac{\partial m(\theta)}{\partial \theta_l} \right]. \end{aligned} \quad (10)$$

Let

$$d_k(\theta) = W^* \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right] \quad (N \times 1). \quad (11)$$

The  $n$ th entry of  $d_k(\theta)$  is the discrete Fourier transform of  $\partial m(t, \theta)/\partial \theta_k$  at  $\omega_n$ , i.e.,

$$d_k(e^{i\omega_n}, \theta) = \text{DFT}_{\omega_n} \left\{ \frac{\partial m(t, \theta)}{\partial \theta_k} \right\}. \quad (11a)$$

Then, for large  $N$

$$\begin{aligned} & \left[ \frac{\partial m(\theta)}{\partial \theta_k} \right]^T R^{-1}(\theta) \left[ \frac{\partial m(\theta)}{\partial \theta_l} \right] \\ &= \sum_{n=1}^N \frac{1}{\phi(\omega_n, \theta)} d_k^T(e^{-i\omega_n}, \theta) d_l(e^{i\omega_n}, \theta) \\ &= \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi(\omega, \theta)} d_k^T(e^{-i\omega}, \theta) d_l(e^{i\omega}, \theta) d\omega. \end{aligned} \quad (12)$$

Inserting (12) and (6) into (3) now yields the desired asymptotic frequency domain expression for the FIM entries:

$$\begin{aligned} [J(\theta)]_{kl} &= \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi(\omega, \theta)} d_k^T(e^{-i\omega}, \theta) d_l(e^{i\omega}, \theta) d\omega \\ &+ \frac{N}{4\pi} \int_{-\pi}^{\pi} \frac{\frac{\partial \phi(\omega, \theta)}{\partial \theta_k} \frac{\partial \phi(\omega, \theta)}{\partial \theta_l}}{\phi^2(\omega, \theta)} d\omega. \end{aligned} \quad (13)$$

Extension of (13) to the complex data case appears in Appendix B.

In some applications (see, e.g., later), it is desirable to have the analogous continuous time version of the FIM (13). Let  $t'$  and  $\omega'$  denote the continuous time (in seconds) and corresponding radial frequency (in radians per second) variables, and let  $\omega'_s$  denote the radial sampling frequency. Assuming that  $\omega'_s$  is larger than the Nyquist frequency, then

$$d_k(e^{i\omega}, \theta) = \text{DFT}_{\omega} \left\{ \frac{\partial m(t, \theta)}{\partial \theta_k} \right\} \approx \frac{1}{\sqrt{N}} \frac{\omega'_s}{2\pi} \mathfrak{F} \left\{ \frac{\partial m(t', \theta)}{\partial \theta_k} \right\} \quad (14a)$$

where  $\mathcal{F}\{\cdot\}$  denotes the continuous time Fourier transform at  $\omega^c = (\omega_s^c/2\pi)\omega$  and

$$\phi(\omega, \theta) = \frac{\omega_s^c}{2\pi} \phi^c(\omega^c, \theta) \quad (14b)$$

where  $\phi^c(\cdot)$  denotes the continuous time power spectrum at  $\omega^c = (\omega_s^c/2\pi)\omega$ .

Using (14a) and (14b) and changing the integration variables from  $\omega$  to  $\omega^c$ , we obtain the asymptotic continuous time analog of (13):

$$\begin{aligned} [J(\theta)]_{kl} &= \frac{1}{2\pi} \int_{-\omega_s^c/2}^{\omega_s^c/2} \frac{1}{\phi^c(\omega^c, \theta)} \mathcal{F}^* \left\{ \frac{\partial m(r', \theta)}{\partial \theta_k} \right\} \mathcal{F} \left\{ \frac{\partial m(r', \theta)}{\partial \theta_l} \right\} d\omega^c \\ &+ \frac{T}{4\pi} \int_{-\omega_s^c/2}^{\omega_s^c/2} \frac{\frac{\partial \phi^c(\omega^c, \theta)}{\partial \theta_k} \frac{\partial \phi^c(\omega^c, \theta)}{\partial \theta_l}}{[\phi^c(\omega^c, \theta)]^2} d\omega^c \end{aligned} \quad (15)$$

where  $T$  denotes the observation time length.

### III. EXAMPLES

#### A. System Identification and Deconvolution

Consider the model of a system's output in noise

$$y(t) = G(q, \alpha) u(t, \beta) + v(t) \quad t = 1, 2, \dots, N \quad (16)$$

where  $G$  is the system's transfer function in the unit delay operator  $q^{-1}$  (i.e.,  $q^{-1}y(t) = y(t-1)$ ) and  $u$  is a deterministic input. It is assumed that  $G$  and  $u$  are known functions of unknown finite-dimensional parameter vectors  $\alpha$  and  $\beta$ . Thus, in this case

$$m(t, \rho) = G(q, \alpha) u(t, \beta); \quad \rho = [\alpha^T \beta^T]^T. \quad (16a)$$

It is also assumed that there is available a parametrization of the unknown noise spectrum  $\phi(\omega, \eta)$  where  $\eta$  denotes an unknown finite-dimensional parameter vector (e.g., an autoregressive moving-average (ARMA) coefficient vector). In the special case where  $u(t, \beta)$  is not a function of  $\beta$  or the input is measurable, the model (16) coincides with conventional system identification models (such as those of [11] and [12]). In the more general case where  $u(t, \beta)$  is a function of  $\beta$  and is not measurable, the model (16) becomes useful for deconvolution problems in geophysics, speech processing, and elsewhere (see, e.g., [15]).

Since the signal and noise in the model (16) are independently parametrized, the FIM has a block diagonal form as in (5), with

$$J_m(\rho) = \begin{bmatrix} J_G & J_{G\eta} \\ J_{\eta G} & J_\eta \end{bmatrix}. \quad (17)$$

Expression (13) is now applicable to the model (16) with

$$d_k(e^{i\omega}, \rho) = \begin{cases} \text{DFT}_\omega \left\{ \frac{\partial}{\partial \alpha_k} G(q, \alpha) u(t, \beta) \right\} = \frac{\partial G(e^{i\omega}, \alpha)}{\partial \alpha_k} U(e^{i\omega}, \beta), & 1 \leq k \leq n_\alpha \\ \text{DFT}_\omega \left\{ \frac{\partial}{\partial \beta_k} G(q, \alpha) u(t, \beta) \right\} = G(e^{i\omega}, \alpha) \frac{\partial U(e^{i\omega}, \beta)}{\partial \beta_k} & n_\alpha < k \leq n_\alpha + n_\beta \end{cases} \quad (18)$$

where  $n_\alpha$  and  $n_\beta$  are the dimensions of  $\alpha$  and  $\beta$ . Inserting (18) into (13) and writing the result in matrix form, we obtain

$$J_G = \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{|U(e^{i\omega}, \beta)|^2}{\phi(\omega, \eta)} G'_\alpha(e^{-i\omega}, \alpha) [G'_\alpha(e^{i\omega}, \alpha)]^T d\omega \quad (19a)$$

$$J_\eta = \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{|G(e^{i\omega}, \alpha)|^2}{\phi(\omega, \eta)} U'_\beta(e^{-i\omega}, \beta) [U'_\beta(e^{i\omega}, \beta)]^T d\omega \quad (19b)$$

$$J_{G\eta} = \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi(\omega, \eta)} G(e^{i\omega}, \alpha) U(e^{-i\omega}, \beta) G'_\alpha(e^{-i\omega}, \alpha) \cdot [U'_\beta(e^{i\omega}, \beta)]^T d\omega \quad (19c)$$

where  $G'_\alpha(e^{i\omega}, \alpha) = \partial G(e^{i\omega}, \alpha) / \partial \alpha$ , etc., and  $J_{\eta G} = J_{G\eta}^*$ . The superscript  $*$  denotes complex conjugate transpose.

In the special case where  $u(t, \beta)$  is measurable or not a function of  $\beta$ ,  $J_m$  reduces to  $J_G$ . Then (19a) coincides with (14.28) of [11] for independently parametrized signal and noise models. Note, however, that (19) is applicable to unknown inputs and was derived here in a different way than in [11].

The CRB of the noise parameters can be found independently of the signal parameters using either the time domain (3) or the frequency domain (6) CRB formulas.

#### B. Time Delay Estimation in Colored Noise

Before we start this example, we note that the following analysis involves only continuous time variables, and therefore, to simplify the notation, we will drop the superscript  $c$ .

A typical estimation problem in active sonar and radar is the following. Let

$$y(t) = gs(t - \tau) + v(t) \quad t \in (0, T) \quad (20)$$

where  $s(t)$  is a known deterministic signal and  $v(t)$  is a zero-mean Gaussian random noise. The unknown signal parameters are the scaling factor  $g$  and time delay  $\tau$ . It is assumed that there is available a parametrization of the noise spectrum  $\phi(\omega, \eta)$  where  $\eta$  denotes an  $(r \times 1)$  unknown parameter vector (e.g., ARMA coefficient vector). Thus, here

$$\theta = \underbrace{[g, \tau, \eta_1, \dots, \eta_r]}_{\rho'} \quad (21)$$

Since the signal and noise in the model (20) are independently parametrized, the FIM has a block diagonal form:

$$J(\theta) = \begin{bmatrix} J_s(\rho) & 0 \\ 0 & J_r(\eta) \end{bmatrix} \quad ((r+2) \times (r+2)) \quad (22)$$

where  $J_s$  is the FIM of the signal parameter vector, i.e.,

$$J_s(\rho) = \begin{bmatrix} J_g & J_{g\tau} \\ J_{\tau g} & J_\tau \end{bmatrix} \quad (2 \times 2) \quad (22a)$$

and  $J_r$  is the FIM of the noise.

The entries of  $J_s$  can now be found from (15):

$$\begin{aligned} [J_s(\rho)]_{kl} &= \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \frac{1}{\phi(\omega, \eta)} \mathcal{F}^* \left\{ \frac{\partial gs(t - \tau)}{\partial \theta_k} \right\} \\ &\cdot \mathcal{F} \left\{ \frac{\partial gs(t - \tau)}{\partial \theta_l} \right\} d\omega \quad k, l = 1, 2. \end{aligned} \quad (23)$$

Thus

$$J_g = \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \frac{|S(\omega)|^2}{\phi(\omega, \eta)} d\omega \quad (24)$$

where  $S(\omega) = \mathcal{F}\{s(t)\}$  and

$$\begin{aligned} J_\tau &= \frac{g^2}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \frac{1}{\phi(\omega, \eta)} \left| \mathcal{F} \left\{ \frac{\partial s(t - \tau)}{\partial \tau} \right\} \right|^2 d\omega \\ &= \frac{g^2}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} \omega^2 \frac{|S(\omega)|^2}{\phi(\omega, \eta)} d\omega. \end{aligned} \quad (25)$$

The cross terms of  $J_s$  are given by

$$J_{g\tau} = J_{\tau g}^* = \frac{g}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} i\omega \frac{|S(\omega)|^2}{\phi(\omega, \eta)} d\omega = 0. \quad (26)$$

The fact that the scaling and delay parameters are decoupled (26) together with the fact that the signal and noise are decoupled (22), implies that the CRB for the delay and scaling are given, respectively, by the inverses of (24) and (25). The CRB of the noise parameters can then be found independently of the signal parameters either by the time domain (3) or frequency domain (6) CRB formulas.

It should be noted that the results (24)–(26) are not surprising. For white noise, these results are well known (see, e.g., [16]). For colored noise, if the noise spectrum  $\phi(\omega, \eta)$  were known, we could get the CRB from the white noise CRB by prewhitening of the signal spectrum which would then give (24)–(26). This same result holds if the noise spectrum is unknown, since the noise and signal are decoupled, and this provides an alternative way of getting (24)–(26).

#### IV. CONCLUSION

The asymptotic frequency domain expression for the Fisher information matrix was derived for Gaussian processes with information in their mean or deterministic component. The result extends the well-known Whittle's formula [1] which considers only the case of information in the covariance (or spectrum). The asymptotic frequency domain expression was derived under the assumption that the observation time is large compared to the correlation time of the random component of the data. Examples were given for system identification, deconvolution, and time delay estimation in colored noise.

#### APPENDIX A A PROOF OF (6)

In this Appendix, for completeness, we prove Whittle's formula (6).

Using the same notation as in Sections I and II, we have

$$\begin{aligned} & \text{tr} \left\{ R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_k} R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_l} \right\} \\ &= \text{tr} \left\{ \left[ W^* R^{-1}(\theta) W \right] \left[ W^* \frac{\partial R(\theta)}{\partial \theta_k} W \right] \left[ W^* R^{-1}(\theta) W \right] \right. \\ & \quad \left. \cdot \left[ W^* \frac{\partial R(\theta)}{\partial \theta_l} W \right] \right\}. \end{aligned} \quad (\text{A.1})$$

From (9) we have, asymptotically

$$W^* \frac{\partial R(\theta)}{\partial \theta_k} W = \text{diag} \left\{ \frac{\partial \phi(\omega_n, \theta)}{\partial \theta_k} \right\} \quad (\text{A.2})$$

substituting (9) and (A.2) into (A.1) we get, asymptotically

$$\text{tr} \left\{ R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_k} R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_l} \right\} = \sum_{n=1}^N \frac{\frac{\partial \phi(\omega_n, \theta)}{\partial \theta_k} \frac{\partial \phi(\omega_n, \theta)}{\partial \theta_l}}{\phi^2(\omega_n, \theta)}. \quad (\text{A.3})$$

Replacing the sum on the right-hand side of (A.3) by an integral, we obtain (6) for large  $N$ .

#### APPENDIX B EXTENSION TO COMPLETE DATA

For complex Gaussian data, the FIM elements are given by [17]

$$\begin{aligned} [J(\theta)]_{kl} &= 2 \text{Re} \left\{ \left| \frac{\partial m(\theta)}{\partial \theta_k} \right|^* R^{-1}(\theta) \left| \frac{\partial m(\theta)}{\partial \theta_l} \right| \right\} \\ &+ \text{tr} \left\{ R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_k} R^{-1}(\theta) \frac{\partial R(\theta)}{\partial \theta_l} \right\} \end{aligned} \quad (\text{B.1})$$

where  $\theta$  is a real-valued parameter vector and  $\text{Re} \{ \}$  denotes the real part of a complex variable. Using similar derivations as in Section II, it is easy to show that the asymptotic frequency domain FIM entries for the complex Gaussian data are

$$\begin{aligned} [J(\theta)]_{kl} &= \text{Re} \left\{ \frac{N}{\pi} \int_{-\pi}^{\pi} \frac{1}{\phi(\omega, \theta)} d_k^*(e^{-i\omega}, \theta) d_l(e^{i\omega}, \theta) d\omega \right\} \\ &+ \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{\frac{\partial \phi(\omega, \theta)}{\partial \theta_k} \frac{\partial \phi(\omega, \theta)}{\partial \theta_l}}{\phi^2(\omega, \theta)} d\omega. \end{aligned} \quad (\text{B.2})$$

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