Voigt function, we propose an analytically tractable family which includes the cosine bell and logistic profiles as special cases. Take

$$\sigma(\omega) = \frac{1}{2\omega_B} cn\left(\frac{\omega}{\omega_B}, k\right) dn\left(\frac{\omega}{\omega_B}, k\right) \quad (19)$$

so that

$$\dot{\varphi}^{-1}(t) = sn(t, k) \quad (20)$$

where cn, dn, and sn are the Jacobian elliptic functions with modulus $0 \leq k \leq 1$. Then [8, p. 18]

$$\varphi(t) = \int sn^{-1}(t, k)\, dt = t\, sn^{-1}(t, k) - \frac{1}{k}\log\{\sqrt{1 - k^2t^2} - k\sqrt{1 - t^2}\}. \quad (21)$$

Consistent with the cosine bell profile, we define (19) over one half period of the elliptic cosine, and so the profile is defined for $|\omega| \leq \omega_B K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. Since [9, 16.6] $cn(t, 0) = \cos t$, $dn(t, 0) = 1$, $K(0) = \pi/2$ and $cn(t, 1) = \text{sech}\, t$, $dn(t, 1) = \text{sech}\, t$, $K(1) = \infty$, it will be seen that (19) properly includes the cosine bell and logistic profiles, as they are written in Table I, at $k = 0$ and $k = 1$, respectively. For intermediate values of k, the elliptic distribution (19) describes appropriately intermediate profiles. An interesting feature of this distribution is that it provides a smooth transition between profiles of finite and infinite bandwidth.

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Asymptotic Cramér-Rao Bounds for Estimation of the Parameters of Damped Sine Waves in Noise

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***Abstract*—The problem of estimating the parameters of a signal composed of several damped sine waves in noise has applications, for example, in transient signal analysis. In this correspondence, explicit, closed form expressions are derived for the Cramér-Rao bound (CRB) of this problem in the case of low damping, low spectral interference between the signal components, and a sufficiently large number of samples. Several conclusions are drawn from the expressions.**

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I. Introduction

Several important problems in signal processing rest upon the estimation of parameters of superimposed sinusoidal signals in noise. This correspondence deals with damped sine waves in noise. The discussion is based on the following model that describes superimposed, real, damped sine waves in noise:

$$y(t) = \sum_{k=1}^{n} \alpha_k e^{-\beta_k t} \sin(\omega_k t + \varphi_k) + e(t) \qquad t = 1, \cdots, N. \quad (1.1)$$

In (1.1), $e(t)$ is an additive disturbance, $y(t)$ is the noisy (scalar) measurement and the unknown parameter vector is given by

$$\theta = [\alpha_1\ \beta_1\ \omega_1\ \varphi_1\ \alpha_2 \cdots \varphi_n]^T. \quad (1.2)$$

The model (1.1) describes transient phenomena occurring, for example, in seismic and passive sonar signal processing. It can also describe the noise corrupted impulse response of a finite dimensional linear system.

The purpose of this correspondence is to study the Cramér-Rao bound (CRB) on θ for the model (1.1), (1.2). An explicit, closed-form expression is derived for the CRB under the assumptions of low damping, low spectral interference between the signal components and a sufficiently large number of samples. The case of complex, superimposed sine waves in noise is also treated.

Previous literature related to the problem considered here can be found in [1]-[6]. Rife and Boorstyn [1] derived the CRB for superimposed, undamped sine waves in white noise. That result was extended to harmonic signals in noise in [2], and to the case of colored noise in [3]. Algorithms for estimation of damped, sinusoidal signals in noise have been described, for example, in [4]-[6].

In Section II of this correspondence, the assumptions are stated and the CRB is derived. The conclusions are summarized in Section III.

II. Assumptions and the CRB

Introduce the notation

$$\Delta\omega_{kl} = \min(|\omega_k - \omega_l|, |\omega_k + \omega_l|) \quad \text{for } k, l = 1, \cdots, n, \quad k \neq l.$$

The assumptions on the signal model (1.1), (1.2) are as follows:

- A1 $Ee(t) = 0$, $Ee(t)\,e(s) = \sigma^2\delta_{t,s}$ and $e(t)$ is Gaussian distributed.
- A2 N is such that $N^2 e^{-\beta_k N} \ll 1$ for $k = 1, \cdots, n$.
- A3 $\omega_k \in [-\pi + \epsilon, \pi - \epsilon]$ for $k = 1, \cdots, n$ and for some small $\epsilon > 0$.
- A4 $|\beta_k/\omega_k| \ll 1$ for $k = 1, \cdots, n$.
- A5 $|(\beta_k + \beta_l)/\Delta\omega_{kl}| \ll 1$ for $k, l = 1, \cdots, n$, $k \neq l$.

Assumption A2 means that the signal is measured until it has faded away, i.e., the number of samples should be sufficiently high. A3 ensures that ω_k is less than the Nyquist frequency for all k. A4 and A5 express the assumptions of low damping and low spectral interference between the signal components. The interpretation of A5 is explained in Appendix A.

Next, the CRB is derived when A1-A5 are valid. The log like-

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lihood function for the estimation problem (1.1), (1.2) is

$$\log p(Y|\theta) = -\frac{N}{2}\log(2\pi) - N\log(\sigma) - \frac{1}{2\sigma^2}\sum_{t=1}^{N}\left[y(t) - \sum_{k=1}^{n}\alpha_k e^{-\beta_k t}\sin(\omega_k t + \varphi_k)\right]^2. \tag{2.1}$$

Differentiation of (2.1) with respect to the elements of θ gives

$$\frac{\partial \log p(Y|\theta)}{\partial \alpha_k} = \frac{1}{\sigma^2}\sum_{t=1}^{N} e^{-\beta_k t}\sin(\omega_k t + \varphi_k)e(t)$$

$$\frac{\partial \log p(Y|\theta)}{\partial \beta_k} = -\frac{1}{\sigma^2}\sum_{t=1}^{N} \alpha_k t e^{-\beta_k t}\sin(\omega_k t + \varphi_k)e(t)$$

$$\frac{\partial \log p(Y|\theta)}{\partial \omega_k} = \frac{1}{\sigma^2}\sum_{t=1}^{N} \alpha_k t e^{-\beta_k t}\cos(\omega_k t + \varphi_k)e(t)$$

$$\frac{\partial \log p(Y|\theta)}{\partial \varphi_k} = \frac{1}{\sigma^2}\sum_{t=1}^{N} \alpha_k e^{-\beta_k t}\cos(\omega_k t + \varphi_k)e(t). \tag{2.2}$$

The following expression results from (2.2) and the definition of the CRB (see, for example, [7]):

$$\mathrm{CRB}^{-1} = E\left[\left(\frac{\partial \log p(Y|\theta)}{\partial\theta}\right)^T\left(\frac{\partial \log p(Y|\theta)}{\partial\theta}\right)\right] = \frac{1}{\sigma^4}\begin{pmatrix}\tau^{11} & \cdots & \tau^{1n}\\ \vdots & \ddots & \\ \tau^{n1} & \cdots & \tau^{nn}\end{pmatrix} \tag{2.3}$$

τ^{kl} are 4×4 matrix blocks whose elements are the expectations of products between the derivatives (2.2). The following lemma will be useful to calculate the τ^{kl}.

Lemma: Assume that the real numbers b, w, c and the positive integer N fulfill

L1: $N^2 e^{-bN} \ll 1$.
L2: $w \in [-2\pi + \epsilon, 2\pi - \epsilon]$ for some small $\epsilon > 0$.
L3: If $w = 0$ then $|b| \ll 1$.
L4: If $w \neq 0$ then $|b/w| \ll 1$.

Then the following results hold true:

$$\sum_{t=1}^{N} e^{-bt} = \frac{1}{b} + O(1) \tag{2.4a}$$

$$\sum_{t=1}^{N} te^{-bt} = \frac{1}{b^2} + O(1) \tag{2.4b}$$

$$\sum_{t=1}^{N} t^2e^{-bt} = \frac{2}{b^3} + O(b) \tag{2.4c}$$

$$\left|\sum_{t=1}^{N} e^{-bt+i(wt+c)}\right| = O\left(\frac{1}{|w|}\right) \quad w \neq 0 \tag{2.4d}$$

$$\left|\sum_{t=1}^{N} te^{-bt+i(wt+c)}\right| = O\left(\frac{1}{|w|^2}\right) \quad w \neq 0 \tag{2.4e}$$

$$\left|\sum_{t=1}^{N} t^2e^{-bt+i(wt+c)}\right| = O\left(\frac{1}{|w|^3}\right) \quad w \neq 0. \tag{2.4f}$$

Proof: See Appendix B. □

The 16 elements of τ^{kl} can now be calculated. From (2.2) it is clear that the 11-element of τ^{kl} is

$$\begin{aligned}\tau_{11}^{kl} &= \sigma^2\sum_{t=1}^{N} e^{-(\beta_k+\beta_l)t}\sin(\omega_k t + \varphi_k)\sin(\omega_l t + \varphi_l)\\ &= \frac{\sigma^2}{4}\sum_{t=1}^{N}\exp\{-(\beta_k+\beta_l)t + i[(\omega_k-\omega_l)t + \varphi_k - \varphi_l]\}\\ &+ \frac{\sigma^2}{4}\sum_{t=1}^{N}\exp\{-(\beta_k+\beta_l)t - i[(\omega_k-\omega_l)t + \varphi_k - \varphi_l]\}\\ &- \frac{\sigma^2}{4}\sum_{t=1}^{N}\exp\{-(\beta_k+\beta_l)t + i[(\omega_k+\omega_l)t + \varphi_k + \varphi_l]\}\\ &- \frac{\sigma^2}{4}\sum_{t=1}^{N}\exp\{-(\beta_k+\beta_l)t - i[(\omega_k+\omega_l)t + \varphi_k + \varphi_l]\}.\end{aligned}$$

Assumptions A2–A5 imply the applicability of the lemma for the sums above. In the case where $k = l$ (2.4a) and (2.4d) are used, and in the case where $k \neq l$ (2.4d) is used in order to conclude that

$$\tau_{11}^{kk} = \sigma^2\left[\frac{1}{4\beta_k} + O(|\omega_k|^{-1})\right]$$

$$\tau_{11}^{kl} = \sigma^2 O(|\Delta\omega_{kl}|^{-1}).$$

The rest of the elements are calculated similarly. Inserting the result in (2.3) gives

$$\mathrm{CRB}^{-1} = \frac{1}{\sigma^2}\,\mathrm{blockdiag}_{k=1,\cdots,n}\begin{bmatrix}\frac{1}{4\beta_k} & -\frac{\alpha_k}{8\beta_k^2} & 0 & 0\\ -\frac{\alpha_k}{8\beta_k^2} & \frac{\alpha_k^2}{8\beta_k^3} & 0 & 0\\ 0 & 0 & \frac{\alpha_k^2}{8\beta_k^3} & \frac{\alpha_k^2}{8\beta_k^2}\\ 0 & 0 & \frac{\alpha_k^2}{8\beta_k^2} & \frac{\alpha_k^2}{4\beta_k}\end{bmatrix} + \frac{1}{\sigma^2}\begin{pmatrix}O^{11} & \cdots & O^{1n}\\ \vdots & \ddots & \vdots\\ O^{n1} & \cdots & O^{nn}\end{pmatrix} \tag{2.5}$$

where O^{kl} are 4×4 matrix blocks given by

$$O^{kk} = \begin{pmatrix}O(|\omega_k|^{-1}) & \alpha_k O(|\omega_k|^{-2}) & \alpha_k O(|\omega_k|^{-2}) & \alpha_k O(|\omega_k|^{-1})\\ \alpha_k O(|\omega_k|^{-2}) & \alpha_k^2 O(|\omega_k|^{-3}) & \alpha_k^2 O(|\omega_k|^{-3}) & \alpha_k^2 O(|\omega_k|^{-2})\\ \alpha_k O(|\omega_k|^{-2}) & \alpha_k^2 O(|\omega_k|^{-3}) & \alpha_k^2 O(|\omega_k|^{-3}) & \alpha_k^2 O(|\omega_k|^{-2})\\ \alpha_k O(|\omega_k|^{-1}) & \alpha_k^2 O(|\omega_k|^{-2}) & \alpha_k^2 O(|\omega_k|^{-2}) & \alpha_k^2 O(|\omega_k|^{-1})\end{pmatrix}$$

$$O^{kl} = \begin{pmatrix}O(|\Delta\omega_{kl}|^{-1}) & \alpha_l O(|\Delta\omega_{kl}|^{-2}) & \alpha_l O(|\Delta\omega_{kl}|^{-2}) & \alpha_l O(|\Delta\omega_{kl}|^{-1})\\ \alpha_k O(|\Delta\omega_{kl}|^{-2}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-3}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-3}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-2})\\ \alpha_k O(|\Delta\omega_{kl}|^{-2}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-3}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-3}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-2})\\ \alpha_k O(|\Delta\omega_{kl}|^{-1}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-2}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-2}) & \alpha_k\alpha_l O(|\Delta\omega_{kl}|^{-1})\end{pmatrix}.$$

A rearrangement of (2.5) results in

$$\mathrm{CRB}^{-1} = \frac{1}{\sigma^2} A(D + A^{-1}OA^{-1})A \tag{2.6}$$

where the matrices A, D, and O are given by

$$A = \mathrm{blockdiag}_{k=1,\cdots,n} \begin{pmatrix} \beta_k^{-1/2} & 0 & 0 & 0 \\ 0 & \alpha_k\beta_k^{-3/2} & 0 & 0 \\ 0 & 0 & \alpha_k\beta_k^{-3/2} & 0 \\ 0 & 0 & 0 & \alpha_k\beta_k^{-1/2} \end{pmatrix}$$

$$D = \mathrm{blockdiag}_{k=1,\cdots,n} \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

$$O = \begin{pmatrix} O^{11} & \cdots & O^{1n} \\ \vdots & \ddots & \\ O^{n1} & \cdots & O^{nn} \end{pmatrix}.$$

An examination of the elements in the second term within parenthesis in (2.6) shows that all these elements are much smaller than 1, in the limit where A4 and A5 hold. Therefore, they can be neglected and the following result holds true.

Theorem: Assuming that A1–A5 hold, then the CRB for the estimation problem (1.1), (1.2) is given by

$$\mathrm{CRB} = \sigma^2\, \mathrm{blockdiag}_{k=1,\cdots,n} \begin{bmatrix} 8\beta_k & \frac{8\beta_k^2}{\alpha_k} & 0 & 0 \\ \frac{8\beta_k^2}{\alpha_k} & \frac{16\beta_k^3}{\alpha_k^2} & 0 & 0 \\ 0 & 0 & \frac{16\beta_k^3}{\alpha_k^2} & -\frac{8\beta_k^2}{\alpha_k^2} \\ 0 & 0 & -\frac{8\beta_k^2}{\alpha_k^2} & \frac{8\beta_k}{\alpha_k^2} \end{bmatrix} \tag{2.7}$$

□

Remark: A similar analysis can be performed for the complex valued signal model

$$y(t) = \sum_{k=1}^{n} \alpha_k \exp(-\beta_k t + i\omega_k t + i\varphi_k) + e(t)$$
$$t = 1, \cdots, N$$
$$\theta = [\alpha_1\ \beta_1\ \omega_1\ \varphi_1\ \alpha_2 \cdots \varphi_n]^T \tag{2.8}$$

where $e(t)$ is complex, white Gaussian noise. With this modification of the assumption A1, the CRB for the estimation problem (2.8) is given by the Theorem, with the bound (2.7) divided by 4.

III. Conclusions

The Cramér–Rao bound has been derived for the problem of estimating the parameters of superimposed, damped sine waves in white Gaussian noise in the limit of low damping, low spectral interference between the signal components and a sufficiently large number of samples. It is only when a large number of samples is used and the damping is low that the CRB is applicable, since a quickly decaying signal with high damping factors generally results in biased estimates. This bias is likely to dominate the estimation error.

Some further comments on the result (2.7) can be given. The obtained expression for the CRB is block diagonal, which means that the achievable estimation accuracies for different signal components are decoupled. The bounds on the estimation accuracy for amplitude (α_k) and damping factor (β_k) are decoupled from the bounds corresponding to frequency (ω_k) and phase (φ_k). This is intuitively clear, since amplitude and damping factor control the size of the signal, while frequency and phase control the oscillation.

The achievable estimation accuracy is much better for damping factor and frequency ($\sim\beta_k^3$) than for amplitude and phase ($\sim\beta_k$). The explanation is that damping factor and frequency are multiplied with the time in the signal model. Small errors in these parameters therefore give rise to signal model errors that grow when time increases. The estimation problem is consequently much more sensitive to errors in damping factor and frequency than to errors in amplitude and phase.

The constant $1/\beta_k$ can be thought of as the effective duration of the kth damped sine wave component. This interpretation clarifies the change from what used to be N in the information matrix of the undamped sine wave parameters in [1] and [8], to $1/\beta_k$ in the present case. Furthermore, observe that (2.7) is independent of N. This is due to the assumption A2 that the signal is measured until it has faded away.

When the β_k tend to zero, also the CRB approaches zero. This is consistent with the undamped case, where the CRB tends to zero when N approaches infinity. The correspondence is explained by A2, since $\beta_k \to 0$ then implies $N \to \infty$.

Appendix A
An Interpretation of the Assumption A5

For reasons of simplicity, consider the continuous time signal

$$y_k(t) = \alpha_k e^{-\beta_k t} \sin(\omega_k t + \varphi_k).$$

Using the Fourier transform of this signal, it is straightforward to establish that the bandwidth is equal to $2\beta_k$, provided that A4 holds. Two spectral peaks corresponding to the frequencies ω_k and ω_l will therefore be well separated when

$$|\omega_k - \omega_l| \gg \beta_k + \beta_l = |\beta_k + \beta_l|$$

which implies

$$\left|\frac{\beta_k + \beta_l}{\omega_k - \omega_l}\right| \ll 1.$$

Since A3 allows negative frequencies as well as positive, A5 is formulated as

$$\left|\frac{\beta_k + \beta_l}{\Delta\omega_{k,l}}\right| \ll 1.$$

Appendix B
Proof of the Lemma

The geometric series in (2.4d) is easily summed

$$\sum_{t=1}^{N} \exp[-bt + i(wt + c)] = \exp[-b + i(w + c)] \frac{1 - \exp[(-b + iw)N]}{1 - \exp(-b + iw)}. \tag{B.1}$$

The two sums in (2.4e), (2.4f) can then be evaluated by differentiation of (B.1) with respect to b. The fact that $N^2 e^{-bN} \ll 1$ is also used. This implies that $Ne^{-bN} \ll 1$ and that $e^{-bN} \ll 1$. The expressions obtained for the sums (2.4d)–(2.4f) then simplify to

$$\sum_{t=1}^{N} \exp\left[-bt + i(wt + c)\right] = \frac{\exp\left[-b + i(w + c)\right]}{1 - \exp\left[-b + iw\right]} \quad (B.2)$$

$$\sum_{t=1}^{N} t \exp\left[-bt + i(wt + c)\right] = \frac{\exp\left[-b + i(w + c)\right]}{\left(1 - \exp\left(-b + iw\right)\right)^2} \quad (B.3)$$

$$\sum_{t=1}^{N} t^2 \exp\left[-bt + i(wt + c)\right] = \frac{\exp\left[-b + i(w + c)\right] + \exp\left[-2b + i(2w + c)\right]}{\left(1 - \exp\left(-b + iw\right)\right)^3}. \quad (B.4)$$

The cases where $w = 0$, $c = 0$, and where $w \neq 0$ are treated separately.

When $w = 0$ and $c = 0$, (B.2) becomes

$$\sum_{t=1}^{N} e^{-bt} = \frac{e^{-b}}{1 - e^{-b}}$$

and since $b| \ll 1$ when $w = 0$, a Taylor series expansion gives (2.4a). The equations (2.4b), (2.4c) are proved similarly.

When $w \neq 0$, only estimates of the sums are required. Beginning with (B.2), the following equality results:

$$\left|\sum_{t=1}^{N} \exp\left[-bt + i(wt + c)\right]\right| = \frac{e^{-b}}{\sqrt{1 + e^{-2b} - 2e^{-b}\cos(w)}}.$$

Since $w \in [-2\pi + \epsilon, 2\pi - \epsilon]$ by L2, the above expression is bounded by a constant depending only on ϵ except when w approaches zero. When w is close to zero also b is close to zero by L4 and a Taylor series expansion gives

$$1 + e^{-2b} - 2e^{-b}\cos(w) \simeq b^2 + w^2 \simeq w^2.$$

It can therefore be concluded that

$$\left|\sum_{t=1}^{N} \exp\left[-bt + i(wt + c)\right]\right| = O\left(\frac{1}{|w|}\right)$$

which is (2.4d). Equations (2.4e)–(2.4f) are proved similarly. This completes the proof of the Lemma.

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Single-Modulus RNS Implementation of Wigner-Ville Time-Varying Spectral Estimations

JoEllen Wilbur and Fred J. Taylor

Abstract—**In this correspondence, a modular arithmetic system, called the single modulus quadratic residue number system (SM-QRNS), is used to implement a fast, high-resolution complex multiply intensive spectral estimator based on the Wigner distribution (WD). The presented SM-QRNS DWD processor is shown to achieve a high real-time bandwidth.**

I. Introduction

The time-frequency representation of signal is a field of growing interest. The traditional time-frequency tool is the short-term discrete Fourier transform (STDFT). An alternative, known as the discrete Wigner distribution (DWD), is gaining increased recognition as being superior to the STDFT in highly dynamic time-varying spectral environments [1]. A popular implementation of the DWD is the pseudo-DWD, denoted simply pDWD or DWD, and is defined to be

$$W(t, k) = \sum_{n=-N/2+1}^{N/2-1} x(t + n)x * (t - n)\left|h(n)\right|^2 e^{-j2\pi nk/(N-1)} \quad (1)$$

where $h(n)$ is a specified window function. Processing of an analytic signal has been shown to have certain advantages in DWO processing [2]. Regardless of the DWD version, it can be seen that the algorithm is *complex multiply intensive* since it can be seen that intrinsic to the DWD is a DFT operating on a windowed inner product kernel. Therefore, for the DWD to be a viable instrument for real-time applications, it must be implemented with a processor capable of performing complex arithmetic at very high data rates.

II. Complex Arithmetic Data Processing

Traditional methods of performing complex arithmetic are both hardware intensive and suffer from long latency. The modular arithmetic system referred to as the residue number system (RNS) has been actively researched as an alternative [3]. The application of residue number systems to high-speed complex airthmetic data processing is now well established [6], [7], [10]. An alternative is the quadratic residue numbering system (QRNS) [4], [5]. In the QRNS, integers are isomorphically mapped to a two-tuple representation of the form $(z, z^*) = ((a + jb) \bmod p, (a - jb) \bmod p)$; $j^2 = -1 \bmod p$, $p = 2^n + 1$ where $j \in Z_p$ and is a quadratic

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