Some practice problems for the final exam

1. Are the following statements correct? A simple YES or NO will do, but read the statements carefully.
   (a) Every subset of \( \mathbb{R} \) is a Borel set.
   (b) If \( X \) and \( Y \) are random variables that have an exponential distribution, then their sum \( X + Y \) also has an exponential distribution.
   (c) If \((X, Y)\) is a jointly Gaussian random vector, then \( X \) and \( Y \) are independent if and only if they are uncorrelated.
   (d) If \((X, Y)\) is a jointly Gaussian random vector, then the best linear estimate \( E\{X|Y\} \) for \( X \) in terms of \( Y \) is also the best overall (i.e., least squares) estimate for \( X \) in terms of \( Y \).
   (e) According to the central limit theorem, any large number of identically distributed random variables is approximately Gaussian.
   (f) The Wiener process has stationary and independent increments.
   (g) The covariance function of a stochastic process is positive definite.
   (h) White noise is a Gaussian stochastic process.
   (i) A zero mean Gaussian stochastic process is stationary if and only its covariance function \( R(s, t) \) is only a function of the difference of times, i.e., \( R(s, t) = \hat{R}(t - s) \).
   (j) Two wide sense stationary processes which have the same mean and covariance are equal.

2. Let \( X \) and \( Y \) be independent and uniformly distributed random variables over the interval \([0, 1]\). For a randomly chosen quadratic polynomial of the form \( P(t) = t^2 + 2tX + Y \), what is the probability that both roots are real?

3. Let \( X \) and \( Y \) be independent scalar Gaussian random variables with mean 0 and variance 1 and let \( R \) and \( \Theta \) denote the corresponding polar coordinates, i.e. \( X = R \cos \Theta \) and \( Y = R \sin \Theta \). Compute the conditional expectations \( E[X^2|\Theta] \), \( E[XY|\Theta] \) and \( E[Y^2|\Theta] \).

4. Let \((X, Y)\) be a jointly continuous random vector with mean \( \mu \) and covariance matrix \( R \) given by

   \[
   \mu = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.
   \]

   Define a new random variable \( Z \) as \( Z = 5X + 2Y + 3 \). Is \( Z \) Gaussian? What are the mean and variance of \( Z \)? Justify your statements.

5. Let \((X, Y, Z)\) be a random vector with mean \( \mu \) and covariance matrix \( R \) given by

   \[
   \mu = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 9 & 3 \\ 2 & 3 & 4 \end{pmatrix}.
   \]

   Find the best linear estimate for \( X \) in terms of the measurements \( y = 1 \) and \( z = 0 \) for \( Y \) and \( Z \).

6. Let \( N_t \) be a Poisson process with intensity \( \lambda > 0 \). For times \( 0 < t_1 < t_2 < t_3 \) and non-negative integers \( m_1, m_2 \) and \( m_3 \) calculate the joint distribution

   \[
P(N(t_1) = m_1, \ N(t_2) = m_2, \ N(t_3) = m_3) \]
7. Compute the covariance function of a Poisson process with intensity \( \lambda > 0 \).

8. Let \( A \) and \( B \) be independent Gaussian random variables with mean 0 and variance 1 and define

\[
X_t = R \cos(2\pi t + \phi)
\]

where

\[
R = \sqrt{A^2 + B^2} \quad \text{and} \quad \phi = \arctan \left( \frac{B}{A} \right).
\]

Show that \( X_t \) is a stationary Gaussian process and calculate its mean \( \mu \) and covariance function \( R \).

9. Let \( W_t = W(t), t \geq 0, \) be a standard (one-dimensional) Wiener process. Define a new stochastic process \( X_t = X(t), t > 0, \) as

\[
X(t) = tW \left( \frac{1}{t} \right)
\]

and set \( X(0) = 0 \). Using the strong law of large numbers it can be shown (and you do NOT need to prove this) that

\[
\lim_{t \to 0} tW \left( \frac{1}{t} \right) = 0.
\]

Use this fact to show that \( X_t \) also is a standard Wiener process.

10. Let \( W_t, t \geq 0, \) be a standard Wiener process. The reflected Brownian motion is the stochastic process defined by \( X_t = |W_t| \). Compute the mean and variance of \( X_t \). Is \( X_t \) a Markov process? If so, what is its transition function?

11. Let \( X_t, t \in \mathbb{R}, \) be a zero mean wide sense stationary process. For a constant \( h > 0 \) define a new process \( Y_t, t \in \mathbb{R}, \) as

\[
Y_t = \frac{1}{h} \int_{t-h}^{t} X_s ds.
\]

Compute the mean \( \mu \) and express the covariance function \( R_Y \) of the stochastic process \( Y_t, t \in \mathbb{R} \) in terms of the covariance function \( R_X \) of the stochastic process \( X_t, t \in \mathbb{R} \). Is \( Y_t \) wide sense stationary?

12. Let \( W_t = W(t), t \geq 0, \) be a standard Wiener process and let \( g = g(t) \) be a continuous, deterministic function defined on \( [0, \infty) \). Denote by

\[
Y_t = \int_{0}^{t} g(s) dW(s)
\]

the stochastic process that is obtained when “white noise” \( dW \) is scaled by the function \( g = g(t) \) and then integrated. Argue that \( Y_t \) is a zero-mean Gaussian process with covariance function

\[
R(s, t) = \int_{0}^{\min(s, t)} g^2(u) du
\]

Hint: Use formal computations involving the Dirac \( \delta \).