Some Practice Problems for the Midterm

1. Define the concepts which make up a probability space \((Ω, ℱ, P)\) and explain what it means for a map \(X : Ω → ℝ\) to be a random variable. Give an example of a probability space \((Ω, ℱ, P)\) and a map \(X\) that is NOT a random variable.

2. Consider the roll of a die and let \(A\) denote the event that an even number comes out on top and let \(B\) be the event that a prime number comes out on top. (F.Y.I., 1 is not a prime number). What is the \(σ\)-algebra on \(Ω\) generated by \(A\) and \(B\)?

3. John and Sue take turns rolling a die. The first one rolling a 6 wins. If John has the first turn, what is the probability that Sue will win?

4. Dichromy is a moderately severe form of color blindness that is present in 2.4% of males, but only in 0.03% of females. According to the last US census, in the United States 50.7% of the population is female. What is the probability that a random selected individual in the US that has dichromy is female?

5. Let \(X\) be a continuous random variable with distribution function \(F\) and density \(f\). Define a new random variable \(Y\) as \(Y = F(X)\). Show that \(Y\) is uniformly distributed in the interval \([0, 1]\).

6. Let \(X_1, \ldots, X_n\) be independent and identically distributed random variables with distribution function \(F\). Define new random variables \(Y\) and \(Z\) as
   \[ Y = \max\{X_1, \ldots, X_n\} \quad \text{and} \quad Z = \min\{X_1, \ldots, X_n\}. \]
   Express the distribution functions of \(Y\) and \(Z\) in terms of the distribution function \(F\).

7. Suppose a current of \(I\) amperes flowing through a resistor of \(R\) ohms varies according to a probability distribution with density
   \[ f(i) = \begin{cases} 
   6i(1 - i) & \text{for } 0 < i < 1, \\
   0 & \text{elsewhere.}
   \end{cases} \]
   If the resistance varies independently of the current according to a probability distribution with density
   \[ g(r) = \begin{cases} 
   2r & \text{for } 0 < r < 1, \\
   0 & \text{elsewhere,}
   \end{cases} \]
   what is the probability that the power \(W, W = I^2R\), is less than \(\frac{1}{2}\)?

8. let \(X\) be a non-negative continuous random variable with mean \(μ\). Show that
   \[ μ = \int_0^∞ P(X ≥ t)dt. \]
   Hint: Express \(P(X ≥ t)\) in terms of the density \(f\) of \(X\).

9. Let \(X\) and \(Y\) be independent random variables which are uniformly distributed over the interval \([0, 1]\). Find the density of the random variable \(Z = X^2 + Y^2\).