# Minimum Mean Square Error Equalization using Priors for Two-Dimensional Intersymbol Interference

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**Abstract**

Joint iterative equalization and decoding schemes are proposed for two-dimensional intersymbol interference channels. Equalization is performed using the minimum mean squared error (MMSE) criterion. Low-density parity-check (LDPC) codes are used for error correction. The equalizer and decoder exchange extrinsic information. The information provided by the LDPC decoder is used to recalculate the filter coefficients at each iteration. This time-varying equalizer has similar performance to schemes based on message-passing and has lower complexity. Time-invariant implementations of the MMSE equalizer are also proposed. Empirical results, extrinsic information transfer (EXIT) charts and bit error rate versus signal to noise ratio curves, show that large savings in computational cost can be achieved at negligible loss of performance using these time-invariant schemes.

## 1 Introduction

Data storage systems have relied primarily on designs based on storing data on tracks. On magnetic media, data is encoded and stored as flux reversals on tracks, with decoding being based on standard algorithms such as the Viterbi or BCJR algorithm. As data densities increase, fundamental limits for recording on tracks are approached, and alternative data storage technologies must be considered. Significant advances have been made in the magnetic recording community in the development of patterned media. The underlying medium is patterned into a two-dimensional array of isolated regions of magnetizations or "islands." Each of these islands is made up of a few grains and is capable of sustaining its magnetization at volumes well below what is predicted by the superparamagnetic limit for conventional magnetic recording. This potentially enables storage densities up to 6 Tb/in$^2$ which is far beyond what can be achieved with conventional recording. Due to the two-dimensional nature of recording each island gets intereference from all its nearest neighbors during readback resulting in two-dimensional (2D) intersymbol interference (ISI).

2D ISI is also a major limiting factor in volume optical storage [1]. In conventional magnetic recording, as the intertrack spacing is reduced the intertrack interference becomes significant enough so that it can no longer be neglected compared to the downtrack ISI. The crosstrack and downtrack interference together give rise to 2D ISI. Even in optical disc storage, advancements in optics enable disc storage with smaller pits. This translates into increased storage density but at the cost of increased crosstrack interference. 2D ISI is harder to deal with than one-dimensional ISI. Standard algorithms, like the Viterbi algorithm, which have been applied with success to one-dimensional ISI become computationally unmanageable for 2D. So there is a need for low complexity algorithms capable of protecting data integrity for systems with 2D ISI.

We describe several approaches to joint equalization and decoding for such systems. These approaches use the minimum mean squared error (MMSE) criterion for equalization in conjunction with low-density parity-check (LDPC) codes for error correction. The potential of 2D MMSE equalization for page-oriented optical memories was demonstrated by Chugg *et al.* [1], [2]. However, the equalizers they proposed were not iterative. Singla *et al.* [3] used iterative 2D MMSE equalization with LDPC decoding and showed significant improvement in performance can be obtained by iterating information between the equalizer and decoder. The equalizer in [3] is spatially invariant: at each iteration the MMSE filter

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is calculated using the entire data and the information obtained from the LDPC decoder. In this paper we propose MMSE equalizers that are spatially and temporally varying. The MMSE equalizer and the LDPC decoder exchange extrinsic information and use it as *a priori* information for the next iteration. This is also known as the principle of turbo equalization.

The MMSE equalization schemes are the extension of those proposed by Tüchler *et al.* [4], [5] for one-dimensional ISI to 2D ISI. Recently, joint equalization and decoding schemes for 2D ISI based on message-passing have been proposed by O'Sullivan *et al.* [3], [6]. These schemes perform message-passing on the joint graph of the error-correction code and the channel ISI. The decoding schemes based on MMSE equalization have lower complexity than the message-passing schemes but have comparable performance. The rest of the paper is organized as follows. Section 2 describes the model we use for the systems with 2D ISI. In Section 3, we describe three MMSE equalization schemes using a linear equalizer. Also described in Section 3 are iterative decoding algorithms using LDPC codes with the equalizers. Results are provided in Section 4 and conclusions in Section 5.

# 2 System Model

The recording system can be represented by the discrete time system shown in Fig. 1. Here $\mathbf{a}$ is the uncoded user data, the channel input, $\mathbf{X}$, is the encoded data, and $\mathbf{R}$ is the channel output. $\mathbf{X}$ is a matrix, typically rectangular, with elements $x(i,j) \in \{\pm 1\}$. The noise, $\mathbf{W}$, is assumed to be additive white Gaussian noise (AWGN) with variance $\sigma_w^2$. The channel output, $\mathbf{R}$, is a matrix with elements

$$r(i,j) = \sum_{l_1=0}^{L-1} \sum_{l_2=0}^{L-1} x(i-l_1, j-l_2) h(l_1,l_2) + w(i,j). \quad (1)$$

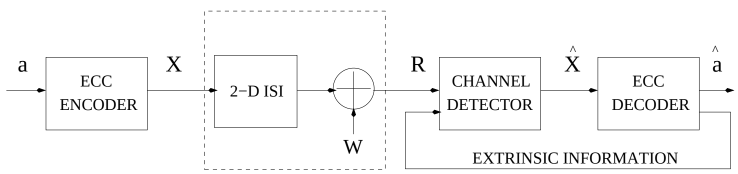



Figure 1: Discrete-time system.

In (1) $L$ represents the number of elements over which the ISI extends in each dimension and $\mathbf{h} = \{h(l_1,l_2)\}_{l_1,l_2=0}^{L-1}$ is the 2D channel point spread function. We assume that the ISI coefficients $h(i,j) \in \mathcal{R}$. For error correction we use LDPC coset codes [7] with the code graph chosen uniformly at random from the ensemble of regular graphs. For the purpose of illustration of our concepts and simulations we use the following point spread function:

$$h = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{pmatrix}.$$

# 3 MMSE Equalization

In this section we describe three MMSE equalization schemes using a linear equalizer. The first equalizer, described in subsection 3.1, is the exact implementation which employs a spatially and temporally varying equalizer. In subsections 3.2 and 3.3 we describe time-invariant approximations of the exact equalizer. In subsection 3.4 we describe joint iterative equalization and decoding schemes using these equalizers with the LDPC decoder.

## 3.1 Exact Implementation

The MMSE equalizer minimizes the mean squared error $E[|x(i,j) - \hat{x}(i,j)|^2]$ between the desired data $\mathbf{X}$ and the filter output $\hat{\mathbf{X}}$. Define $vec(\mathbf{A})$ as the operation that rasters the matrix $\mathbf{A}$ (either row-wise or column-wise) into a column vector. The linear MMSE estimate of $x(i,j)$ is,

$$\begin{aligned} \hat{x}(i,j) &= E[x(i,j)] + \sum_{l_1=-N}^{N} \sum_{l_2=-N}^{N} \Big( r(i-l_1, j-l_2) - E[r(i-l_1, j-l_2)] \Big) c(i,j:l_1,l_2) \\ &= m(i,j) + \mathbf{c}_{ij}^T (\mathbf{r}_{ij} - E[\mathbf{r}_{ij}]). \quad (2) \end{aligned}$$

Here $\mathbf{r}_{ij}$ is the $(2N+1)^2 \times 1$ vector $vec(\{r(i-l_1, j-l_2)\}_{l_1,l_2=-N}^{N})$. $(\{c(i,j:l_1,l_2)\}_{l_1,l_2=-N}^{N})$ are the MMSE filter coefficients for bit $x(i,j)$ and $\mathbf{c}_{ij}$ is the vector $vec(\{c(i,j:l_1,l_2)\}_{l_1,l_2=-N}^{N})$. The superscript $T$ denotes matrix transposition. $m(i,j) = E[x(i,j)]$ is the mean of $x(i,j)$. The mean, $m(i,j)$, and variance, $v(i,j)$, of $x(i,j)$ can be calculated using the *a priori* information. Specifically, if $L_a(x(i,j))$ is the *a priori* log-likelihood ratio (LLR) of $x(i,j)$, then,

$$\begin{aligned} m(i,j) &= tanh(L_a(x(i,j))/2) \\ v(i,j) &= 1-(m(i,j))^2. \end{aligned}$$

The coefficients of the Wiener filter are obtained by solving the Wiener-Hopf equations

$$\mathbf{K_{rr}}\mathbf{c}_{ij} = \mathbf{K}_{\mathbf{r}x}, \tag{3}$$

where $\mathbf{K_{rr}} = E[\mathbf{r}_{ij}\mathbf{r}_{ij}^T]$ and $\mathbf{K}_{\mathbf{r}x} = E[\mathbf{r}_{ij}x(i,j)]$. This solution results in an estimate $\hat{x}(i,j)$ of $x(i,j)$ which is unbiased, i.e., $E[\hat{x}(i,j)] = E[x(i,j)]$.
For the system model in (1),

$$\begin{aligned} \mathbf{K_{rr}} &= \sigma_w^2\mathbf{I}_{(2N+1)^2} + \mathbf{H}\mathbf{V}_{ij}\mathbf{H}^T && (4) \\ \mathbf{K}_{\mathbf{r}x} &= v(i,j)\mathbf{s}^T && (5) \\ E[\mathbf{r}_{ij}] &= \mathbf{H}\mathbf{m}_{ij}, && (6) \end{aligned}$$

where $\mathbf{I}_{(2N+1)^2}$ is the $(2N+1)^2 \times (2N+1)^2$ identity matrix, $\mathbf{H}$ is the $(2N+1)^2 \times (2N+L)^2$ channel convolution matrix. $\mathbf{V}_{ij}$ =diag$[vec(\{v(i-l_1, j-l_2)\}_{l_1,l_2=-2N-L}^{2N+L})]$, where diag$[\mathbf{a}]$ is a diagonal matrix with the elements of vector $\mathbf{a}$ on the main diagonal. $\mathbf{s} = \mathbf{H}[\mathbf{0}_{1\times(N+L-1)(2N+1+L)}\ 1\ \mathbf{0}_{1\times N(2N+1+L)}]^T$. Finally, $\mathbf{m}_{ij}$ =$vec(\{m(i-l_1, j-l_2)\}_{l_1,l_2=-2N-L}^{2N+L})$. Using (3)-(5) the filter coefficients for the exact implementation can be calculated as

$$\mathbf{c}_{EX} = v(i,j)(\sigma_w^2\mathbf{I}_{(2N+1)^2} + \mathbf{H}\mathbf{V}_{ij}\mathbf{H}^T)^{-1}\mathbf{s}. \tag{7}$$

The filter output, $\hat{x}(i,j)$ can be calculated using (2), (6), (7),

$$\hat{x}(i,j) = m(i,j) + v(i,j)\mathbf{s}^T(\sigma_w^2\mathbf{I}_{(2N+1)^2} + \mathbf{H}\mathbf{V}_{ij}\mathbf{H}^T)^{-1}(\mathbf{r}_{ij} - \mathbf{H}\mathbf{m}_{ij}). \tag{8}$$

The computation of the filter coefficients involves inversion of a $(2N+1)^2 \times (2N+1)^2$ matrix which causes a high computational load. One approximation which can be used to reduce this load is to have time-invariant coefficients. In the following we discuss two time-invariant implementations of the exact MMSE equalizer [4].

## 3.2 Approximate Implementation I: No Prior Information

If we assume that $L_a(x(i,j)) = 0, \forall i,j$, i.e., no *a priori* information is provided to the equalizer, then we obtain a time-invariant vector of filter coefficients

$$\mathbf{c}_{NA} = (\sigma_w^2\mathbf{I}_{(2N+1)^2} + \mathbf{H}\mathbf{H}^T)^{-1}\mathbf{s}. \tag{9}$$

## 3.3 Approximate Implementation II: Perfect Prior Information

If we assume perfect *a priori* information, i.e., $|L_a(x(i,j))| \rightarrow \infty, \forall i,j$ then we obtain another time-invariant filter

$$\mathbf{c}_{MF} = \frac{\mathbf{s}}{\sigma_w^2 + \mathbf{s}^T\mathbf{s}}. \tag{10}$$

### 3.4 Iterative MMSE Equalization and Decoding

In this subsection we describe how the MMSE equalizers are used with the LDPC decoder to perform joint equalization and decoding. The equalizers use the extrinsic information $L_D(x(i,j))$, provided by the LDPC decoder, to compute the filter coefficients and the MMSE estimate. During these calculations for a particular bit $x(i,j)$, the LLR $L_D(x(i,j))$ is set to 0. This is to ensure that only extrinsic information is passed from the equalizer to the decoder. This modifies the calculations of the filter coefficients for the exact MMSE equalizer in (7)

$$\mathbf{c}_{EX} = (\sigma_w^2 \mathbf{I}_{(2N+1)^2} + \mathbf{H}\mathbf{V}_{ij}\mathbf{H}^T + (1 - v(i,j))\mathbf{s}\mathbf{s}^T)^{-1}\mathbf{s}. \tag{11}$$

and the MMSE estimate in (2)

$$\hat{x}(i,j) = \mathbf{c}_{ij}^T(\mathbf{r}_{ij} - \mathbf{H}\mathbf{m}_{ij} + m(i,j)\mathbf{s}). \tag{12}$$

As in [4], it is assumed that after MMSE equalization the probability density functions (pdf) $p(\hat{x}(i,j)|x(i,j) = x), x \in \{\pm 1\}$, are Gaussian with parameters $\mu_{ij}(x) = E[\hat{x}(i,j)|x(i,j) = x]$ and $\sigma_{ij}^2(x) = Cov(\hat{x}(i,j), \hat{x}(i,j)|x(i,j) = x)$. Under this assumption the output LLR

$$L_E(x(i,j)) = \log \frac{p(\hat{x}(i,j)|x(i,j) = +1)}{p(\hat{x}(i,j)|x(i,j) = -1)}$$

becomes

$$L_E(x(i,j)) = \frac{2\hat{x}(i,j)\mu_{ij}(+1)}{\sigma_{ij}^2(+1)}. \tag{13}$$

The statistics $\mu_{ij}(x)$ and $\sigma_{ij}^2(x)$ can be calculated using the filter coefficients

$$\mu_{ij}(x) = x \cdot \mathbf{c}_{ij}^T \mathbf{s} \tag{14}$$

$$\sigma_{ij}^2(x) = \mathbf{c}_{ij}^T(\sigma_w^2 \mathbf{I}_{(2N+1)^2} + \mathbf{H}\mathbf{V}_{ij}\mathbf{H}^T - v(i,j)\mathbf{s}\mathbf{s}^T)\mathbf{c}_{ij}. \tag{15}$$

The output LLRs can now be calculated using (12)-(15) and the filter coefficients. The LDPC decoder uses these LLRs as *a priori* information and performs a fixed number of message-passing iterations before passing its extrinsic information to the MMSE equalizer.

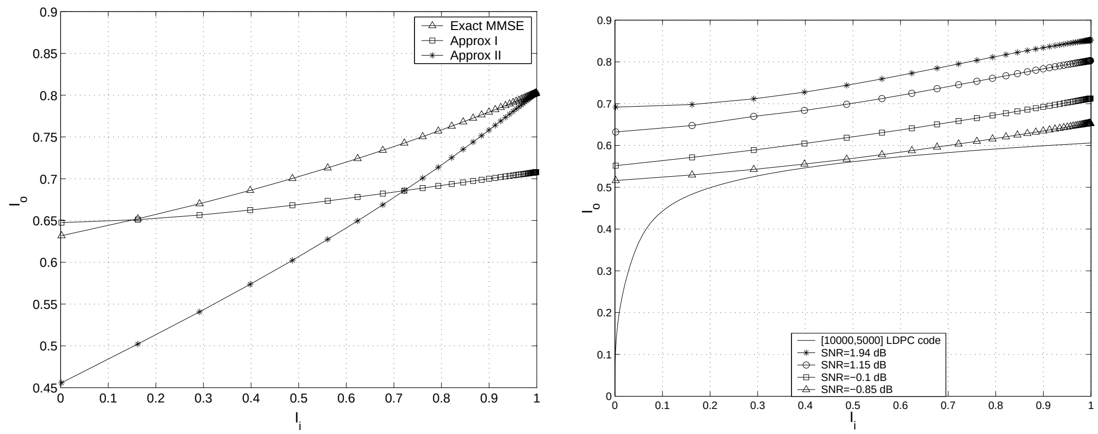
## 4 Results and Discussion

We use extrinsic information transfer (EXIT) charts [8] to compare the performance of the equalizers described in the previous section. EXIT charts show how the "quality" of the output information varies with the quality of the input information for a particular receiver component. For the EXIT analysis, the equalizer is modeled as a device mapping the channel output $\mathbf{R}$ and the *a priori* LLRs $L_i$ to a sequence of output LLRs $L_o$. It is assumed that the sequence $L_i$ is independent and identically distributed (i.i.d.) Gaussian with the pdf specified by a single parameter, $\sigma_i^2$. As proposed by ten Brink [8] the output LLRs, $L_o$, are also reasonably well approximated by a single parameter Gaussian distribution for a second parameter $\sigma_o^2$. For both the pdfs the magnitude of the mean is half the variance. Using ten Brink's approach, we plot $I_o$, the mutual information between $L_o$ and $X$, versus $I_i$, the mutual information between $L_i$ and $X$. Here $X$ is a binary valued random variable taking values +1 or -1 with equal probability. The pdf of $L_o$ is estimated by making a histogram of the LLR values at the equalizer output.

Fig. 2(a) shows the exit charts for the equalization schemes at signal to noise ratio (SNR) 1.15 dB. The SNR is calculated as

$$SNR = 10 \cdot \log_{10} \frac{\sum_{l_1,l_2=0}^{L-1} h^2(l_1,l_2)}{2\sigma_w^2}.$$

The filters each have a 5×5 support. For each EXIT chart, $10^6$ randomly chosen equiprobable symbols $x(i,j) \in \pm 1$ were generated and transmitted over the ISI channel. The corresponding *a priori* LLRs $L(x(i,j))$ are generated for a given $\sigma_i^2$ by randomly generating $10^6$ LLRs with the pdf $N(\sigma_i^2/2, \sigma_i^2)$




(a) (b)

Figure 2: . (a) EXIT charts for the MMSE equalization schemes at 1.15 dB SNR. (b) EXIT charts for the exact MMSE scheme for different SNRs. Also shown is the EXIT chart for a block length 10000, regular (3,6) LDPC code.

and flipping the sign of all LLRs where $x(i, j) = -1$ yielding the pdf $N(-\sigma_i^2/2, \sigma_i^2)$ for those LLRs. As expected the exact MMSE scheme has the best performance. The approximate I scheme has a good starting behavior but poor behavior at high values of $I_i$ whilst the opposite is observed for the approximate II scheme. As observed in [4], the EXIT charts for the two approximate schemes in Fig. 2(a) suggest using a scheme which switches between the two approximate equalizers. This scheme would have both the good starting behavior of the approximate I equalizer and good ending behavior of the approximate II equalizer.

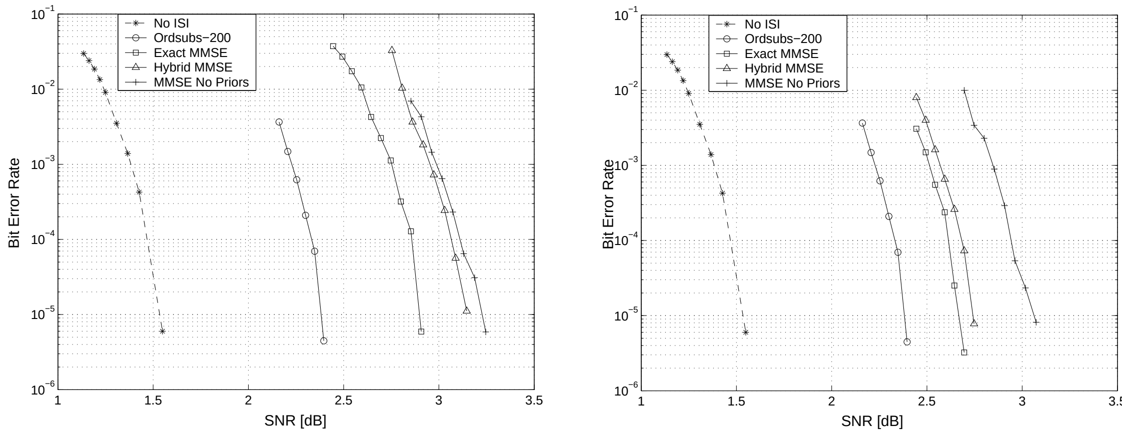
Fig. 2(b) shows the EXIT charts for the exact MMSE equalizer at different SNRs. The filter again has a 5×5 support. Also shown is the EXIT chart for a block length 10000, regular (3,6) LDPC code. As the SNR is reduced the gap between the EXIT charts of the equalizer and decoder becomes narrower until the two touch at an SNR of -0.85 dB. This value of the SNR gives us an idea of how much noise the equalizer-decoder pair can tolerate so as to reliably recover the data. This approach was first proposed by ten Brink [8] for parallel concatenated codes and was used to construct "good" codes.

Fig. 3 shows bit error rate (BER) versus SNR performance for the MMSE equalization schemes. The SNR is calculated as

$$SNR = 10 \cdot \log_{10} \frac{\sum_{l_1, l_2=0}^{L-1} h^2(l_1, l_2)}{2C \cdot \sigma_w^2}$$

where $C$ is the code rate. Fig. 3(a) shows the performance after three iterations of MMSE equalization and Fig. 3(b) shows the performance after ten equalizations. These schemes are compared with the performance of an LDPC code on an AWGN channel with no ISI, which is the first curve from the left. The LDPC code used is again a block length 10000, regular (3,6) code. The second curve from the left is the performance of the ordered subsets message-passing algorithm [6]. This algorithm performs joint equalization and decoding by message-passing on a partitioned graph of the LDPC code and the channel ISI.

The exact MMSE decoding scheme after ten iterations is about 0.2 dB worse than the ordered subsets message-passing algorithm. The hybrid MMSE decoding scheme in Fig. 3 is a scheme that switches between the approximate I and II schemes. The equalizer switches between the two approximate equalizers depending on which equalizer yields a larger value of $I_o$ given $I_i$. $I_o$ is a monotonically increasing function of $\sigma_o^2$ which can be estimated in terms of the statistics $\mu_{ij}(x)$ and $\sigma_{ij}^2(x)$ for both approximate approaches [4]. The hybrid scheme has better performance than the approximate I scheme. Also its performance is very close to the exact MMSE scheme. The performance of the approximate II scheme is not shown since it is very bad. This is due to the poor starting behavior of the approximate II scheme. For the MMSE based decoding schemes, twenty iterations of LDPC decoding are performed after each equalization.




(a) (b)

Figure 3: (a) BER vs SNR curves for the MMSE equalization schemes at third iteration. (b) BER vs SNR curves for the MMSE equalization schemes at tenth iteration.

# 5 Conclusions

We have proposed joint equalization and decoding schemes for two-dimensional intersymbol interference channels. The equalization is based on the minimum mean square error criterion. Three equalization schemes are proposed. The exact MMSE scheme uses a spatially and temporally varying filter and has a high computational cost. Two approximate and less complex schemes employing time-invariant filters are also proposed. These equalizers are then used with LDPC codes to perform joint equalization and decoding. The equalizer and decoder exchange extrinsic information and use it as *a priori* information for the next iteration. The exact MMSE decoding scheme has performance 0.2 dB worse than the ordered subsets message-passing scheme [6]. A lower complexity hybrid scheme which combines the approximate I and II schemes is shown to have performance very close to the exact MMSE decoding scheme.

# 6 Acknowledgments

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