We present an exact dissipation model for correlated photon transport in waveguide QED systems. This model rigorously incorporates the infinitely many degrees of freedom of the full three-dimensional photonic scattering channels in the non-excitable ambient environment. We show that the photon leakages to the scattering channels can be accounted for by a reduced Hamiltonian and a restricted eigen-state, with a resultant atomic dissipation. This model is valid for arbitrary photonic Fock and coherent states. © 2017 Optical Society of America

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Dissipations in cavity and waveguide quantum electrodynamic systems (cQED and wQED) are ubiquitous [1–7] and play an important role in the correlated few-photon transport. Therefore, the mechanisms of dissipations in the quantum regime have acquired the community’s interests for decades. A prevalent approach treats the irreversible dissipation processes as the particle or energy leakages from the system of interest to the external environment due to the coupling between them. The environment, in general, can be characterized into two different types of nature. On one hand, the environment contains an excitable medium and is conventionally treated as a reservoir of oscillators [8,9]. From then on, many techniques have been developed to investigate the consequences of reservoir-induced damping, including the density-matrix method [10], the Monte Carlo wavefunction formalism [11], and the quantum Langevin approach [12]. On the other hand, the environment can be non-excitable and is modeled as photonic scattering channels [13]. Both models describe the photon leakages out of the QED system of interest as dissipations. For the single-photon case, it has been demonstrated in the first scenario that the dynamics of the system can be described by a reduced Hamiltonian (i.e., adding an imaginary part in the atomic transition frequency) and a restricted eigen-state (i.e., omitting the scattering channels of the environment in the wavefunction) which are restricted to the Hilbert space of the QED system only [9]. It is not clear, a priori, whether the correlated multi-photon transport properties can still be described by the same approach.

In this Letter, we investigate the scattering of arbitrary photonic Fock states by an atom in the wQED systems in the presence of dissipations. The ambient environment is treated as a homogeneous, three-dimensional, non-excitable, photonic phase space, with infinitely many photonic scattering channels. Quantum mechanically, the photons in the combined system (QED system + environment) are entangled, and their transport properties can only be obtained by solving the full dynamics involving both the system and the environment. In the following, we show rigorously that the reduced Hamiltonian approach is still valid for arbitrary photonic Fock and coherent states, when the environment contains only the non-excitable medium. Such a scenario applies to a vast majority of current wQED experiments. A direct numerical approach, which incorporates the conventional approximations (the rotating wave approximations and linear dispersion approximations), but otherwise takes into account the infinitely many degrees of freedom of the combined system, is obviously computationally infeasible. Consequently, our exact results (after the aforementioned approximations) provide a tremendously convenient starting point for analytical investigations and numerical models for the few-photon transport case in wQED systems.

Figure 1 schematically shows a general wQED system in a non-excitable environment. An N-photon Fock state \( |N\rangle \) is launched from the left and interacts with the atom. Each photon can escape the waveguide to the environment or remain waveguided. The environment is characterized by an infinite number of three-dimensional scattering channels, each of which is uniquely specified by the direction of the photon momentum, \( \mathbf{n} \), and the polarization, \( s \). The infinite number of degrees of freedom of the scattering channels give rise to an irreversible process macroscopically. To begin with, we first discretize the photonic phase space as follows: the polar angle, \( \theta \), and the azimuthal angle, \( \phi \) (Fig. 1), are discretized uniformly so that now each scattering channel is uniquely specified by a doublet \( \{ \theta_i, \phi_j \} = \{ \frac{\pi j}{M_1}, \frac{\pi i}{M_2} \} \), where \( i = 0, 1, \ldots, M_1; j = 0, 1, \ldots, M_2 \); and \( \sum_n = \sum_{ij} M_1 M_2 \Delta \theta_i \Delta \phi_j / \pi^2 \).

Here, we detail the computational model. Specifically, the Hamiltonian of the wQED system, \( H_S \), is [14]
channel. The orientation within the environmental scattering channels, $\vec{n}_s \parallel \vec{R}$ and $\vec{n}_L = -\vec{n}$, $s$ denotes the polarization. $\theta \in [0, \pi]$ and $\varphi \in [0, \pi]$.}

\[
H_{SC} \hbar = \sum_{\vec{n}_s} \int d\xi (c_{\vec{n}_sL}^\dagger(\xi) c_{\vec{n}_sR}(\xi) + c_{\vec{n}_sR}^\dagger(\xi) c_{\vec{n}_sL}(\xi)) \sigma - \\
+ \sigma + c_{\vec{n}_sL}(\xi) + c_{\vec{n}_sR}^\dagger(\xi) \sigma + \sigma ^\dagger c_{\vec{n}_sR}(\xi),
\]  

where $\nabla_{\vec{n}}$ denotes the coupling strength between the atom and the channel $\vec{n}_s$. The complete Hamiltonian is given by $H = H_S + H_C + H_{SC}$.

We now demonstrate that the reduced Hamiltonian approach is valid in the single-photon case. The most general eigenstate involving both the system and the environment is given by $|\Psi\rangle = \left( \int dx \Phi(x) c_{R}(x) + \phi(x) c_{L}^\dagger(x) \right) \sigma_i$, where $|\Phi(x)\rangle$ denotes the single-photon wavefunction for the right-moving waveguided photon, and $\phi(x)$, $\phi_{\vec{n}_s}$, and $\phi_{\vec{n}_L}$ can be analogously defined. $\epsilon$ is the atomic excitation amplitude. $|0, -\rangle$ is the photonic ground state and $|0, +\rangle$ is the photonic vacuum state with the atom at the ground state. By imposing the condition $H|\Psi\rangle = \epsilon |\Psi\rangle$ (where $\epsilon$ is the eigen-energy of the combined system), we obtain the equations of motion:

\[
\begin{align*}
\epsilon \phi_{\vec{n}_s}(\xi) &= -iv_\xi \partial_x \phi_{\vec{n}_s}(\xi) + \nabla \delta(\xi) \epsilon + \omega_\xi \phi_{\vec{n}_s}(\xi), \\
\epsilon \phi_{\vec{n}_L}(\xi) &= iv_\xi \partial_x \phi_{\vec{n}_L}(\xi) + \nabla \delta(\xi) \epsilon + \omega_\xi \phi_{\vec{n}_L}(\xi), \\
\epsilon \phi_{\vec{n}_s}(\xi) &= -iv_\xi \partial_x \phi_{\vec{n}_R}(\xi) + \nabla \delta(\xi) \epsilon + \omega_\xi \phi_{\vec{n}_R}(\xi), \\
\epsilon \phi_{\vec{n}_L}(\xi) &= iv_\xi \partial_x \phi_{\vec{n}_R}(\xi) + \nabla \delta(\xi) \epsilon + \omega_\xi \phi_{\vec{n}_R}(\xi), \\
e \epsilon &= \omega_\xi \epsilon + \nabla \phi_{\vec{n}_s}(\xi) + \phi_{\vec{n}_L}(\xi) + \sum_{\vec{n}_L} \nabla \phi_{\vec{n}_L}(\xi) + \phi_{\vec{n}_L}(\xi) - \phi_{\vec{n}_R}(\xi) - \phi_{\vec{n}_R}(\xi).
\end{align*}
\]

By expressing $\phi_{\vec{n}_s}(0)$ and $\phi_{\vec{n}_L}(0)$ in terms of $\epsilon$, Eqs. (5a), (5b), and (5e) become self-consistent involving only the system variables. As the photon is initially injected into the waveguide and is scattered into the channels by the atom, the single-photon wavefunction in the scattering channel, $\phi_{\vec{n}_s}$ and $\phi_{\vec{n}_L}$, must take the form of $\phi_{\vec{n}_s}(\xi) = r_{\vec{n}_s} \theta(\xi) e^{i\delta_{\vec{n}_s}}$ and $\phi_{\vec{n}_L}(\xi) = q_{\vec{n}_L} \theta(-\xi) e^{-i\delta_{\vec{n}_L}}$. One now plugs $\phi_{\vec{n}_s}$ and $\phi_{\vec{n}_L}$ into Eqs. (5c) and (5d) to obtain $r_{\vec{n}_s} = q_{\vec{n}_L} = -i \nabla \delta(\xi) \epsilon$. One also has that $\phi_{\vec{n}_s}(0) = r_{\vec{n}_s}/2$, $\phi_{\vec{n}_L}(0) = q_{\vec{n}_L}/2$, where we have employed $\theta(0) = 1/2$. Thus, Eq. (5e) now reads as

\[
\epsilon \epsilon = \omega_{\xi} \epsilon + \nabla \phi_{\vec{n}_s}(\xi) + \phi_{\vec{n}_L}(\xi) + \sum_{\vec{n}_L} \nabla \phi_{\vec{n}_L}(\xi) + \phi_{\vec{n}_L}(\xi) - \phi_{\vec{n}_R}(\xi) - \phi_{\vec{n}_R}(\xi),
\]

which now does not involve the wavefunctions of the scattering channels. Equations (5a), (5b), and (6) now form a set of self-consistent equations which only involve the system variables $\phi_{\vec{n}_s}$, $\phi_{\vec{n}_L}$, and $\epsilon$. Equation (6) states that such an approach is equivalent to adding an imaginary part $-i\gamma_{\vec{n}_s}$ to $\omega_{\vec{n}_s}$ in $H_S$, where $\gamma_{\vec{n}_s} = \sum_{\vec{n}_L} \nabla_{\vec{n}_L}^2\epsilon$ is identified as the atomic dissipation rate. These equations can be derived from a reduced Hamiltonian $H_{SC}$ and a restricted eigenstate $|\Psi\rangle = (\int dx \Phi(x) c_{\vec{n}_s}^\dagger(x) + \phi(x) c_{\vec{n}_L}(x) + \sigma_\epsilon) |0, -\rangle$ that only involves the variables of the system. The dissipative process is described by an imaginary part $-i\gamma_{\vec{n}_s}$. $F = \gamma_{\vec{n}_s}/\gamma$ is the Purcell enhancement factor, where $\gamma$ is the atomic dissipation in free space.

We now generalize the approach for the two-photon case. To facilitate the mathematical description, we transform $H$ to the even $(H_e)$ and odd $(H_o)$ modes $(H = H_e + H_o)$ such that $H_e$ and $H_o$ are decoupled ($[H_e, H_o] = 0$; see [15]). Since the
photon-atom interaction is present in the even mode and not in the odd one, only the even mode is taken into account from now on. $H_e$ takes the following form:

$$H_e = \int dx \phi^*_e(x)(-i\nu_g \partial_x)\phi_e(x) + \int dx V\delta(x)[\phi^*_e(x)\sigma_+ + \sigma_+\phi_e(x)]$$

$$+ \sum_{n,i} \left( \int dx \phi^*_{n,i}(\xi)(-i\nu \partial_\xi)\phi_{n,i}(\xi) \right) + \omega_n a^\dagger_n a_n + \omega_g a^\dagger_g a_g$$

$$+ \sum_{n,i} \int d\xi V_{n,i}(\xi)\phi^*_{n,i}(\xi)\sigma_+ + \sigma_+\phi_{n,i}(\xi),$$

(7)

where $V = \sqrt{2} V_n$, $V_n = \sqrt{2} V_{n,i}$. $H_e$ describes that photons propagate unidirectionally to the $+x$ or $+\xi$ direction. We have used the operator transformations $c^*_e(x) = [c_e(x) + c^\dagger_e(-x)]/\sqrt{2}$, $c^*_n(\xi) = [c_{n,i}(\xi) + c_{n,i}^\dagger(-\xi)]/\sqrt{2}$, and $c_{n,i}(\xi) = [c_{n,i}(\xi) - c_{n,i}^\dagger(-\xi)]/\sqrt{2}$ (see [15]). The corresponding even-mode eigenstate is given by

$$|\Psi\rangle = \left( \int dx_1 dx_2 \frac{1}{\sqrt{2}} [\phi(x_1, x_2) c^*_e(x_1) c^*_e(x_2)$$

$$+ \int dx_1 dx_2 \frac{1}{\sqrt{2}} [\phi_{n,i}(x, \xi) c^*_e(x) c^*_e(x) + \phi_{n,i}(\xi, x) c_{n,i}(\xi) c_{n,i}(x)]$$

$$+ \int dx c^*_e(x)\phi_{n,i}(x, \xi)\sigma_+ + \sum_{n,i} \int d\xi c_{n,i}(\xi)\phi^*_{n,i}(\xi)\sigma_+ + \cdots \right) |0, -\rangle,$$

(8)

where $\phi$ denotes the two-photon wavefunction in the even waveguided mode. $\phi_{n,i}$ denotes the wavefunction where the first (second) photon is in the even waveguided mode and the other photon in the even mode of the channel $n$. $\epsilon$ and $\epsilon_n$ denote the atomic excitation wavefunctions where the photon is in the even waveguided mode and the $n$ channel, respectively. In Eq. (8), we omit the terms involving both photons in the scattering channels as it turns out that such terms do not directly affect the following results. The equations of motion relevant to the analysis are

$$\epsilon \phi(x_1, x_2) = -i\nu_g (\partial_{x_1} + \partial_{x_2})\phi(x_1, x_2)$$

$$+ \frac{1}{\sqrt{2}} [V\delta(x_1)\phi(x_2) + V\delta(x_2)\phi(x_1)] + \omega_g \phi(x_1, x_2),$$

(9a)

$$\epsilon \phi_{n,i}(x, \xi) = -i(\nu \partial_{\xi} + \nu \partial_{x_1})\phi_{n,i}(x, \xi)$$

$$+ \frac{1}{\sqrt{2}} [V_{n,i}\epsilon(\xi)\partial_{\xi} + V_{n,i}\epsilon(x)\partial_{x_1}] + \omega_g \phi_{n,i}(x, \xi),$$

(9b)

$$\epsilon \phi_{n,i}(\xi, x) = -i(\nu \partial_{x_2} + \nu \partial_{\xi})\phi_{n,i}(\xi, x)$$

$$+ \frac{1}{\sqrt{2}} [V_{n,i}\epsilon(\xi)\partial_{x_1} + V_{n,i}\epsilon(x)\partial_{\xi}] + \omega_g \phi_{n,i}(\xi, x),$$

(9c)

$$\epsilon e(x) = \omega_e e(x) - i\nu_g \partial_{x_1} e(x) + \frac{V}{\sqrt{2}} [\phi(x, 0) + \phi(0, x)]$$

$$+ \sum_{n,i} \frac{V_{n,i}}{\sqrt{2}} \phi_{n,i}(x, 0) + \phi_{n,i}(0, x).$$

(9d)

Plugging Eq. (10) into Eq. (9d), one obtains

$$\epsilon e(x) = (\omega_e - i\nu_g \partial_{x_1}) e(x) + \frac{V}{\sqrt{2}} [\phi(x, 0) + \phi(0, x)],$$

(10)

where we have used $\phi_{n,i}(x, 0) = [\phi_{n,i}(x, 0^+) + \phi_{n,i}(x, 0^-)]/2$. In Eq. (10), we have also employed the condition $\phi_{n,i}(x, 0^+) = 0$. $\phi_{n,i}(x, 0^-)$ is the probability amplitude when one photon is at $x$ in the even waveguided mode, and the other photon is at $\xi = 0$ in the even mode of the channel $n$. Since the photon leaks from the atom to the scattering channels at $\xi = 0$, and the photon can only propagate in the $+\xi$ direction, $\phi_{n,i}(x, 0^-)$ vanishes.

Equations (9a) and (11) now form a set of self-consistent equations that describe two-photon transport, which can also be derived by a reduced Hamiltonian and a restricted even-state. This procedure validates the approach in the two-photon case.

We now describe the general arbitrary $N$-photon Fock state $|N\rangle$ case. The equations of motion of relevant variables are

$$\epsilon \phi(x_1, \ldots, x_N) = \omega_e \phi - i\nu_g (\partial_{x_1} + \cdots + \partial_{x_N})\phi$$

$$+ \frac{V}{\sqrt{N}} \sum_i e(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)\delta(x_i),$$

(12a)

$$\epsilon \phi_{n,i}(x_1, \ldots, x_N, \xi, x_{N+1}, \ldots, x_N) = \omega_{n,i} \phi_{n,i}$$

$$- i\nu_g (\partial_{x_1} + \cdots + \nu \partial_{x_1}) + \cdots + \partial_{x_{N+1}})\phi_{n,i}$$

$$+ \frac{1}{\sqrt{N}} [V_{n,i}\epsilon(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)\delta(\xi)]$$

$$+ \sum_{i,i' \neq j} \omega_{n,i} \phi_{n,i'}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)\delta(x_i),$$

(12b)

$$\epsilon \phi(x_1, \ldots, x_N, \xi, x_{N+1}, \ldots, x_N) = \omega_e \epsilon - i\nu_g (\partial_{x_1} + \cdots + \partial_{x_{N+1}})\epsilon$$

$$+ \frac{V}{\sqrt{N}} [\phi(0, x_1, \ldots, x_{N+1}) + \cdots + \phi(x_1, \ldots, x_{N+1}, 0)]$$

$$+ \sum_{n,i} \frac{V_{n,i}}{\sqrt{N}} \phi_{n,i}(x_1, \ldots, x_{N+1}, 0, x_1, \ldots, x_{N+1}),$$

(12c)

where $\phi$ denotes the $N$-photon wavefunction in the even waveguided mode. $\phi_{n,i}$ denotes the $N$-photon wavefunction wherein $i$-th photon is in the even mode of channel $n$, while all others are in the even waveguided mode ($j = 1, 2, 3, \ldots, N$). $\epsilon$ denotes the atomic excitation amplitude when $N - 1$ photons are in the even waveguided mode. $\epsilon_n$ denotes the atomic excitation where the photon arrangement is the same as $\phi_{n,i}$. Similar to Eq. (10), for the $N$-photon case, now we have

$$\phi_{n,i}(x_1, \ldots, x_{i-1}, 0^+, x_j, \ldots, x_{N+1})$$

$$= \phi_{n,i}(x_1, \ldots, x_{j-1}, 0^+, x_j, \ldots, x_{N+1}) - i\frac{V_{n,i}}{\sqrt{N}} \epsilon(x_1, \ldots, x_{N+1}).$$

(13)
Plugging Eq. (13) into Eq. (12c), one obtains

\[ e(x_1, \ldots, x_{N-1}) = \left( \omega - \sum_{n=1}^{N} \frac{\nabla_{n}^{2}}{v} \right) e - iv_{g}(\partial_{x_1} + \cdots + \partial_{x_{N-1}})e \]

\[ + \frac{V}{\sqrt{N}} \left[ \phi(0, x_1, \ldots, x_{N-1}) + \cdots + \phi(x_1, \ldots, x_{N-1}, 0) \right]. \quad (14) \]

Equations (12a) and (14) are now self-consistent to describe the N-photon correlated transport, which can be derived from a reduced Hamiltonian and a restricted eigen-state, thereby validating the approach in the N-photon case. As a coherent state is a linear superposition of Fock states, our dissipation model also applies when the input is a coherent state. We also note that the conclusion remains valid regardless of the explicit form of \[ \nabla_{n}. \]

One now converts the expression of \( \gamma_{n} \) from the discrete phase space to the continuum. Writing that \( \sum_{n} \nabla_{n}/v = \sum_{n} \sum_{i} \nabla_{i}^{2}(\theta, \varphi) \Delta \theta \Delta \varphi / \pi^{2} \nu \) and taking the limit \( M_{1}, M_{2} \to \infty \), one obtains

\[ \gamma_{n} = \sum_{i} \frac{1}{\pi^{2}} \int_{0}^{\pi} d\theta \int_{0}^{\pi} d\varphi \frac{\nabla_{i}^{2}(\theta, \varphi)}{v}. \quad (15) \]

We also note that different discretization schemes only differ in a Jacobian and should lead to the same continuum limit. When the discretization scheme is to discretize the solid angle, then \( \gamma_{n} \) takes the form \( \sum_{i} \int_{0}^{\pi} \int_{0}^{\pi} \sin \theta \sin \varphi \frac{\nabla_{i}^{2}(\theta, \varphi)}{v} d\theta d\varphi \).

We now apply Eq. (15) to approximate \( \nabla_{i} \), for two simple geometries. The first case is a radiating dipole embedded in a uniform medium with an index \( n \). For a quantum oscillator undergoing an exponential decay of energy in the medium, one has \( E(t) = E_{0}e^{-t/t'} \). \( E_{0} = h\omega \) is the energy quantum of the dipole oscillator, where \( \omega \) is the angular frequency. The radiating power within a short period of time, \( \Delta t \), is \( [E(0) - E(\Delta t)] / \Delta t = \gamma_{r}E_{0} \). Equating \( \gamma_{r}E_{0} \) and the radiating power of the oscillator, \( \int_{0}^{\pi} \int_{0}^{\pi} \sin \theta \sin \varphi \frac{\nabla_{i}^{2}(\theta, \varphi)}{v} \) (where \( \sin \theta \sin \varphi / v \) is the time average of the Poynting vector), and one obtains \( \nabla_{i}(\theta, \varphi) = \frac{1}{\nu} \sqrt{\frac{p(\omega \sin \theta)}{c}} \), where \( p \) is the dipole moment; \( \epsilon_{0} \) and \( c \) are the permittivity and the speed of light in vacuum, respectively; \( r \) is the radius of interest. We note that \( \gamma_{n} \) obtained here is exactly the same as the result using the Weiskopf-Wigner theory [16]. Using the explicit form of \( \nabla_{i} \) in Eq. (15), one obtains \( F = n \) in this case.

The second case is a dipole embedded in the center of a slab waveguide with the dipole moment pointing perpendicularly out of plane, as illustrated in Fig. 2(a). The waveguide has an index, \( n_{m} \), and the surrounding environment has an index, \( n \). We assume that the thickness of the waveguide is much larger than the wavelength of radiation so that the dispersion of the waveguide can be considered as linear, and the ray picture applies. In this case, the only emitted light with an incident angle smaller than the critical angle \( \theta_{c} = \arcsin \frac{n_{m}}{n} \) can escape from the waveguide. Following the same procedure by equating the dipole radiating power, \( 2 \int_{0}^{\pi} \int_{0}^{\pi} \sin \theta \sin \phi \nabla_{i}^{2}(\theta, \varphi) \), and \( \gamma_{r}E_{0} = 2h\omega / \pi^{2} \int_{0}^{\pi} \int_{0}^{\pi} \sin \theta \sin \varphi \nabla_{i}^{2}(\theta, \varphi) / v \), one obtains \( \nabla_{i}(\theta, \varphi) \) is given by \( \sqrt{\frac{\cos \theta \sin \phi}{\nu \epsilon_{0} \sin \theta / c^{2}}} \), (the Snell’s law and \( \theta_{m} = \phi \) are used), \( F = \frac{2\cos \theta \sin \phi}{\nu \epsilon_{0} \sin \theta / c^{2}} n_{m} \) can be similarly obtained in this case. For a semiconductor waveguide \( (n_{m} \approx 3) \) sandwiched by oxide layers \( (n \approx 1.5) \), \( F \approx 0.08 \). In general, the values of \( \nabla_{n} \) can only be obtained from numerical simulations.

The atomic dissipation given rise from the photonic scattering channels has been discussed in the literature. Conventionally, photonic scattering channels are alternatively labeled by the folded scattering channel modes \( n_{u} \) and \( n_{d} \), as illustrated in Fig. 2(b) (see [17]). By invoking the transformations \( n_{u}(\xi) = (n_{u}(\xi) + n_{d}(\xi)) / \sqrt{2} \) and \( n_{d}(\xi) = (n_{u}(\xi) - n_{d}(\xi)) / \sqrt{2} \), it is straightforward to show that the two labeling schemes are mathematically equivalent.

Finally, we comment on the cavity dissipations. By employing the explicit photon-cavity interaction [9] and using the same approach outlined above, one can show that the photonic scattering channels also result in a reduced Hamiltonian and a restricted eigen-state. In addition, our approach may also facilitate the investigations of the atom-light interaction in a nano-fiber configuration [18,19].

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