Frames and a vector-valued ambiguity function

John J. Benedetto and Jeffrey J. Donatelli

***Abstract*—The setting is that of vector-valued codes of length N. The background for this setting is our construction of new complex valued constant amplitude zero autocorrelation codes (off dc) – CAZACs, which serve as coefficients for phase coded waveforms with prescribed ambiguity function behavior. Vector-valued CAZACs are relevant in light of vector sensor and MIMO technologies. The goal is to define the discrete vector-valued ambiguity function. Our notion of frame multiplication allows us to make this definition. The theory and relevance of such ambiguity functions is developed, including computation of code from received discrete vector-valued ambiguity function data.**

I. Introduction

A. Problem

Let $\mathbb{C}$ denoted the field of complex numbers, and let $\mathbb{C}^d$ be complex d-dimensional Euclidean space. For a given N-periodic code, u, taking values in $\mathbb{C}^d$, our goal is to define a discrete periodic vector-valued ambiguity function $A_p^d(u)$, which is compatible with the physically grounded ambiguity function $A(w)$ defined on the real line $\mathbb{R}$. For a phase-coded waveform w, u will be the code used to define w. There is applicable rationale for effecting such a definition, e.g., to model vector sensing environments. However, in this paper we shall concentrate on a meaningful mathematical definition inspired by the fact that $A(w)$ is a special case of the classical cross-ambiguity function on $\mathbb{R}$, which, in turn, can be viewed as the short time Fourier transform (STFT) used in spectrogram analysis.

B. Ambiguity function

In 1953, P.M. Woodward [10] defined the *narrow band radar ambiguity function* or, simply, *ambiguity function*, see Equation (I.1). It is a device formulated to describe the effects of range and Doppler on matched filter receivers. Woodward acknowledged the influence of Shannon's communication theory, from 1948, on his ideas; and he explained the relevance of "ambiguity" in radar signal processing, perhaps best conceived in terms of a form of the uncertainty principle.

A function $w : \mathbb{R} \to \mathbb{C}$ is a *finite energy signal* if $\|w\|_2 = \left(\int_{\mathbb{R}} |w(s)|^2 ds\right)^{\frac{1}{2}} < \infty$. In this case, we write $w \in L^2(\mathbb{R})$. The *Fourier transform*, $\widehat{w} : \mathbb{R} \to \mathbb{C}$, of w can be well–defined by the formal expression,

$$\widehat{w}(\gamma) = \int_{\mathbb{R}} w(t) e^{-2\pi i t\gamma} dt, \ \gamma \in \mathbb{R}.$$

Let $\mathbb{R}^2$ be the direct product $\mathbb{R} \times \mathbb{R}$. The *ambiguity function* $A(w)$ of $w \in L^2(\mathbb{R})$ is

$$\begin{aligned} A(w)(t,\gamma) &= \int_{\mathbb{R}} w(s+t)\overline{w(s)} e^{-2\pi i s\gamma} ds \qquad \text{(I.1)} \\ &= e^{\pi i t\gamma} \int_{\mathbb{R}} w(s+\frac{t}{2})\overline{w(s-\frac{t}{2})} e^{-2\pi i s\gamma} ds, \end{aligned}$$

for $(t,\gamma) \in \mathbb{R}^2$. An elementary form of the aforementioned uncertainty principle, apropos the ambiguity function, is the formula,

$$\iint_{\mathbb{R}^2} |A(w)(t,\gamma)|^2 dt d\gamma = \|w\|_2^4, \qquad \text{(I.2)}$$

where $w \in L^2(\mathbb{R})$. Equation (I.2) asserts that $A(w)$ can not be concentrated arbitrarily closely to the origin $(0,0) \in \mathbb{R}^2$. A more refined form of Equation (I.2), which is called the *radar uncertainty principle*, is the following assertion. *If* $\|w\|_2 = 1$, *and if* $X \subseteq \mathbb{R}^2$ *and* $\epsilon > 0$ *have the property that*

$$\iint_X |A(w)(t,\gamma)|^2 dt d\gamma \geq 1 - \epsilon,$$

then $|X| \geq 1 - \epsilon$, *where* $|X|$ *is the area (Lebesgue measure)* of X.

Remark I.1. The *Wigner distribution* $W(w)$ of $w \in L^2(\mathbb{R})$ was introduced by E. Wigner in 1932 in the context of quantum mechanics. It is defined by

$$W(w)(t,\gamma) = \int_{\mathbb{R}} w(t+\frac{s}{2})\overline{w(t-\frac{s}{2})} e^{-2\pi i s\gamma} ds,$$

for $(t,\gamma) \in \mathbb{R}^2$. Up to a rotation, $W(w)$ is the two dimensional Fourier transform of $A(w)$, e.g., see [4].

Remark I.2. If $v, w \in L^2(\mathbb{R})$, the *narrow band cross–*

ambiguity function $A(v,w)$ of v and w is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}ds = e^{2\pi i t\gamma}\int_{\mathbb{R}} v(s)\overline{w(s-t)}e^{-2\pi i s\gamma}ds.$$

Thus, $A(v,w)$ is the *short–time Fourier transform* (STFT) of $v \in L^2(\mathbb{R})$ with *window* w. STFTs are a staple in spectral analysis where one analyzes *spectrograms* $|A(v,w)|$ recorded from various experiments, e.g., in speech analysis. They are also the basis for time–frequency or Gabor or Weyl–Heisenberg analysis.

In the ambiguity function analysis of phase coded waveforms, w, with N lags, and as mentioned in Subsection I-A, it is both interesting and relevant to analyze the *discrete periodic ambiguity function*, $A_p(u)$, of the code, u, used to define w. For the set $\mathbb{Z}$ of integers, the additive group of integers modulo N is denoted by $\mathbb{Z}_N$, and thus $u : \mathbb{Z}_N \to \mathbb{C}$ is N-periodic on $\mathbb{Z}$. In this case, $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ is

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi i kn/N}.$$

We shall not treat the *discrete aperiodic ambiguity function*, $A_a(u)$, in the paper. Also, we hasten to add that direct applicability of $A_p(u)$ in some radar applications is problematic. In fact, normally, the entire Doppler shift of interest lies within an interval of length $2/(Nt_b)$, where t_b is the duration of a phase element, and not only at the discrete values provided by $A_p(u)$. However, an analysis of $A_p(u)$ is a prerequisite for a deeper understanding of $A(u)$ vis a vis waveform design for local ambiguity behavior and for a solution of Bueckner's ambiguity uniqueness problem, see [3], [8], [1], [5].

Our goal is to define $A_p^d : \mathbb{Z}_N \to \mathbb{C}^d$, cf., [9].

C. Frames

Definition I.3. Let H be a separable Hilbert space, e.g., $L^2(\mathbb{R})$ or $\mathbb{C}^d$. A set $F = \{E_j\}_{j\in J} \subseteq H$ is a *frame* for H if

$$\exists A, B > 0 \quad \text{such that} \quad \forall u \in H,$$
$$A\|u\|^2 \leq \sum_{j\in J} |\langle u, E_j\rangle|^2 \leq B\|u\|^2.$$

A frame F is a *tight frame* if we can choose $A = B$. If, in addition, each E_j is unit-norm, we say that F is a *unit-norm tight frame*. A finite unit norm tight frame is referred to as a FUNTF.

The following result is well-known and elementary to verify, e.g., [6] [7].

Theorem I.4. If $\{E_j\}_{j=0}^{N-1}$ is a FUNTF for $\mathbb{C}^d$, then

$$\forall u \in \mathbb{C}^d, \quad u = \frac{d}{N}\sum_{j=0}^{N-1} < u, E_j > E_j.$$

Example I.5. We define *DFT frames* or *complex harmonic frames*. Let $N \geq d$ and form an $N \times d$ matrix using any d columns of the $N \times N$ Discrete Fourier Transform (DFT) matrix $(e^{2\pi i jk/N})_{j,k=0}^{N-1}$. The rows of this $N \times d$ matrix, up to multiplication by $1/\sqrt{d}$, form a FUNTF for $\mathbb{C}^d$.

II. Frame multiplication

Given $u : \mathbb{Z}_N \to \mathbb{C}^d$. If $d = 1$, then we can write $A_p(u)$ as

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} < u(m+k), u(k)e_{nk} >, \tag{II.1}$$

where $e_n = e^{2\pi i n/N}$. For $d > 1$, the problem of defining a discrete periodic ambiguity function has two natural settings: it is $\mathbb{C}-$valued or $\mathbb{C}^d-$valued, i.e., $A_p^1(u)(m,n) \in \mathbb{C}$ or $A_p^d(u)(m,n) \in \mathbb{C}^d$.

In this section we consider the case of A_p^1. Motivated by (II.1), we must find a sequence $\{E_k\} \subseteq \mathbb{C}^d$ and multiplication $*$ so that

$$A_p^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} < u(m+k), u(k)*E_{nk} > \in \mathbb{C} \tag{II.2}$$

is a meaningful ambiguity function.

There is a natural way to define this multiplication motivated by the fact that $e_m e_n = e_{m+n}$. To effect this definition, we shall make the following three *ambiguity function assumptions*. First, we *assume* there is a sequence $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ with the property that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$. Second, to deal with $u(k)*E_{nk}$ in (II.2) where $u(k) \in \mathbb{C}^d$, we also *assume* that $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$. The multiplication nk is modular arithmetic in $\mathbb{Z}_N$. Third, we *assume* that $* : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum\sum c_j d_k E_j * E_k.$$

Given $u, v : \mathbb{Z}_N \to C^d$. With the *ambiguity function assumptions,* we have the following basic calculation for $m, n \in \mathbb{Z}_N$ by Theorem I.4:

$$u(m) * v(n)$$

$$= \frac{d}{N}\sum_{j=0}^{N-1} < u(m), E_j > E_j * \frac{d}{N}\sum_{s=0}^{N-1} < v(n), E_s > E_s$$

$$= \frac{d^2}{N^2}\sum_{j=0}^{N-1}\sum_{s=0}^{N-1} < u(m), E_j >< v(n), E_s > E_j * E_s$$

$$= \frac{d^2}{N^2}\sum_{j=0}^{N-1}\sum_{s=0}^{N-1} < u(m), E_j >< v(n), E_s > E_{j+s}$$

$$\in \mathbb{C}^d.$$

Example II.1. *($A^1_p(u)$ for DFT frames)*

Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions. Then,

$$E_m * E_n = \frac{d^2}{N^2}\sum_{j=0}^{N-1}\sum_{s=0}^{N-1} < E_m, E_j >< E_n, E_s > E_{j+s}. \quad \text{(II.3)}$$

Now assume that $\{E_j\}_{j=0}^{N-1}$ is a DFT frame, and let r designate a fixed column. Assume, without loss of generality, that the $N \times d$ matrix for the frame consists of the first d columns of the $N \times N$ DFT matrix. Then, (II.3) gives

$$(E_m * E_n)(r)$$

$$= \frac{1}{N^2\sqrt{d}}\sum_{j=0}^{N-1}\sum_{s=0}^{N-1}\sum_{t=0}^{d-1}\sum_{k=0}^{d-1} e_{(m-j)t}e_{(n-s)k}e_{(j+s)r},$$

which, in turn, equals

$$\frac{1}{N^2\sqrt{d}}\sum_{t=0}^{d-1}\sum_{k=0}^{d-1} e_{mt+nk}\sum_{j=0}^{N-1} e_{(r-t)j}\sum_{s=0}^{N-1} e_{(r-k)s}$$

$$= \frac{1}{N^2\sqrt{d}}\sum_{t=0}^{d-1}\sum_{k=0}^{d-1} e_{mt+nk}N\delta(r-t)N\delta(r-k)$$

$$= \frac{e_{(m+n)r}}{\sqrt{d}} = E_{m+n}(r).$$

Consequently, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$. In particular, we have shown that $A^1_p(u)$ is well-defined by (II.2) and can be written down explicitly for the case of DFT frames and componentwise multiplication $*$ in $\mathbb{C}^d$.

The definition of $*$ is intrinsically related to the "addition" defined on the indices of the frame elements. In fact, it is not pre-ordained that this "addition" must be modular addition in $\mathbb{Z}_N$, as was the case in Example II.1. Formally, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{Z}_N$. The following example exhibits this phenomenon for the familiar case of cross products from the calculus. In [2], we have developed the theory for finite groups, both abelian and non-abelian.

Example II.2. *($A^1_p(u)$ for cross product frames)*

Define $* : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$. Let $\{i, j, k\}$ be the standard basis for $\mathbb{C}^3$, e.g., $i = (1, 0, 0) \in \mathbb{C}^3$. We have that $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. The union of tight frames and the 0 vector is a tight frame. In fact, $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$. The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \to \mathbb{Z}_7$, where we compute the following: $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc. Thus, the ambiguity function assumptions are valid, with the verification of bilinearity from the definition of the cross product being a tedious calculation. In any case, we can now use frame multiplication to obtain the following formula:

$$u * v = \frac{1}{2^2}\sum_{j=1}^{6}\sum_{s=1}^{6} < u, E_j >< v, E_s > E_{j \bullet s}.$$

Consequently, $A^1_p(u)$ is well-defined by (II.2) for the case of this cross product frame and associated frame multiplication $*$, which is defined in terms of the classical cross product on $\mathbb{C}^3$ with corresponding index "addition" $\bullet$. It is elementary to write the explicit formula for $A^1_p(u)$, and this and its manipulation and generalization are treated in [2].

In Example II.2, the frame was generated by taking the set of all possible products of the basis elements In general this set need not be finite. In the infinite case this set could be a frame which allows us to write the corresponding multiplication as a frame multiplication. In the finite case we can always form a frame as we did in Example II.2. In the process we are using a bilinear multiplication to form an index operation. Alternatively, we might be able to reverse this process and use an index operation to form a bilinear operation on some Hilbert space. Whatever the direction, it is also desirable to know when these operations give rise to tight frames and, in particular, FUNTFs. Further it is natural to consider the indices as a group, and to develop the corresponding frame and frame multiplication from a given group, thereby defining the ambiguity function $A^1_p(u)$. in terms of a given finite group. This is one of the points of view of [2].

When the ambiguity function $A^1_p(u)$ is defined for $u : \mathbb{Z}_N \to \mathbb{C}^d$, we have the following calculation:

$$A^1_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} < u(m+k), u(k) * E_{nk} >$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} < u(m+k), \frac{d}{N}\sum_{j=0}^{N-1} < u(k), E_j > E_j * E_{nk} >$$

$$= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} < E_j, u(k) >< u(m+k), E_{j+nk} > .$$

Intuitively, this formulation makes some sense. When thinking in terms of Doppler shifts, the ambiguity function should correlate the original signal with itself after a phase translation. In the multidimensional case the multiplication defined above seems to be a natural way to perform this translation, where the indices of the harmonic frames represent the phase.

III. Vector-valued ambiguity function

In this section, $\{E_k\}_{k=0}^{N-1}$ will be a DFT frame. We define a vector valued version of the above formulation of the ambiguity function as follows: $A^d_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}^d$ for $u : \mathbb{Z}_N \to \mathbb{C}^d$, where

$$A^d_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}. \quad \text{(III.1)}$$

We shall see that this definition is compatible with that of $A_p(u)$ in II.1, as well as the point of view of defining ambiguity functions as natural special cases of the STFT.

Definition III.1. Given $u : \mathbb{Z}_N \to \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of u is defined by

$$\forall\ n \in \mathbb{Z}_N, \quad F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) mutliplication.

The inversion process for the vector-valued case is analagous to the 1 dimensional case. However, we must use a different multiplication for the frequency domain to avoid divisibility issues. As such it is convenient to take $d << N$ or N prime, reminiscent of Tchebotorov's theorem and recent work of Tao. Without loss of generality assume that the rth element of E_n is $e^{-2\pi i r n/N}$. Consider the vector-valued Fourier transform as a map from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$, where $\ell^2(\mathbb{Z}_N, \omega)$ is the set of complex sequences of length N with weighted bilinear mutliplication $u(*)v = u*v*\omega$, where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$. If N is prime, we also have that the vector-valued Fourier transform is unitary.

Theorem III.2. *(Vector-valued Fourier inversion)* The vector valued Fourier transform is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m)$$

$$= \frac{d}{N}\sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

The usual translation properties of the Fourier transform hold.

Theorem III.3. Given $u : \mathbb{Z}_N \to \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. Define translation by s as $\tau_s u(g) = u(s+g)$ and set $E^s(k) = E_{sk}$. Then,
i. $F(\tau_s u) = E^{-s}\hat{u}$;
ii. $F(E^s u) = \tau_s \hat{u}$.

Let $u, v \in \mathbb{C}^d$ and set $\{u, v\} = u * \overline{v}$ where $*$ is pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$. We compute

$$A^d_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} (\tau_m u(k)) * \overline{F^{-1}(\tau_n \hat{u})(k)},$$

and, notationally, we write the right side as the generalized inner product,

$$\frac{1}{N}\sum_{k=0}^{N-1} \{\tau_m u(k), F^{-1}(\tau_n \hat{u})(k)\},$$

which is compatible with our point of view of defining the vector-valued ambiguity function in the context of the STFT. We have introduced the notation, $\{...,...\}$, because of the general theory developed in [2].

IV. Epilogue

In the sequel [2], we give full proofs of those results whose proofs we have omitted here, and we extend the theory to the cases where indicial addition is generalized to the settings of both finite abelian and non-abelian groups. The ultimate goal is to go beyond discrete ambiguity functions and to characterize the local behavior of analogue ambiguity functions in both the time-frequency narrowband case and the wavelet wideband case. Along the way, there are major problems dealing with waveform design and the associated uniqueness problems of determining waveforms from empirical norm values of received ambiguity data.

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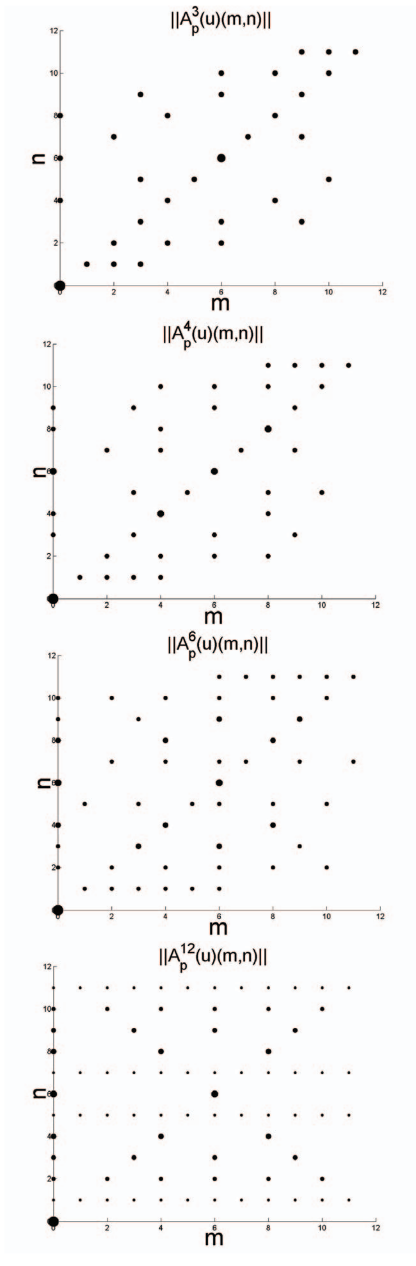

Fig. 1. Intensity of $||A_p^d(u)(m,n)||$ for $d = 3, 4, 6, 12$ where u is a Wiener CAZAC Frame of length 12

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