Instantaneous Frequency Estimation Using Sequential Bayesian Techniques

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Abstract— **The online estimation of the instantaneous frequency (IF) of time-varying (TV) signals with highly nonlinear phase functions is a challenging problem. In this paper, we propose an IF estimation method using Bayesian techniques, which combines particle filtering and Markov Chain Monte Carlo (MCMC) methods, to sequentially estimate highly nonlinear TV frequency variations as piecewise linear functions. Simultaneously applying parameter estimation and model selection, the new technique is extended to the IF estimation of multicomponent signals. Using simulations, we demonstrate the performance of our approach for different signals and environments.**

I. Introduction

Most signals in nature are TV as their spectral characteristics vary with time. This frequency variation with time cannot be obtained using the Fourier transform as this transform simply expands a signal as a linear combination of single frequencies that exist over all time. For a single component TV signal, $s(t) = a(t)e^{j2\pi c\varphi(t)}$, the frequency at a particular time can be described by the IF,

$$\zeta_s(t) = c\frac{d}{dt}\varphi(t). \tag{1}$$

Here, $a(t)$ is the signal's amplitude, $\varphi(t)$ is its phase, and c is its rate of frequency change with respect to time.

Estimating the IF of a signal can often provide information about propagation in an unknown environment, and it has been shown to be a powerful analysis tool in many applications such as underwater acoustics and radar. Thus, IF estimation methods are an important research topic in the literature.

Time-frequency (TF) processing can be used as a non-parametric IF estimation approach. For example, the Wigner distribution (WD), $W_s(t, f)$, of a signal $s(t)$ is a very popular time-frequency representation (TFR) that can display the signal spectrum with high resolution since $\int_{-\infty}^{\infty} f W_s(t, f) df = \zeta_s(t)$ [1]. The IF can also be estimated by extracting the ridges (or peaks) of a TFR and then applying a peak detection technique [2]–[4]. However, this method may not work well with multicomponent signals or signals in low signal-to-noise ratios (SNRs).

IF estimation methods were also developed based on parametric statistical models to describe the signal, and often using maximum likelihood estimation techniques. For example, Newton's method was used to determine the maximum of the log-likelihood function of the signal [5]. Doucet [6] proposed a Bayesian approach to IF estimation by using the MCMC method with parametric non-stationary processes with some prior knowledge. Note, however, that it may be difficult to find the correct mathematical model for use in these methods.

In this paper, we propose an IF estimation method based on Bayesian techniques that uses a particle filter algorithm with the MCMC method, and approximates the IF as a piecewise linear function of IFs of non-overlapping linear frequency-modulated (FM) chirps. The new method is shown to work even at low SNRs, and does not require a closed form model for the phase function $\varphi(t)$ in (1).

The paper is organized as follows. In Section 2, we provide an overview of the sequential estimation approach of static parameters as proposed in [7]. Based on this technique and piecewise linear approximations, we propose our approach of IF estimation in Section 3; the method is also extended to multiple component signal estimation using model selection. Some simulation results are presented in Section 4.

II. Sequential Estimation of Static Parameters

Our proposed IF method is based on sequentially estimating highly nonlinear TV frequency variations as piecewise linear functions; we will accomplish this by sequentially estimating the static parameters of these linear functions. As a result, we first provide an overview of the sequential estimation of the static parameters approach proposed in [7] that is based on the combined use of particle filtering and the MCMC method. We begin by formulating a system model with the unknown parameters to be estimated based on the given measurements.

A. System Models

In order to analyze and make inference about a dynamic system, two models are required at time step k. The process model describes the evolution of the state vector $\mathbf{x}_k$ with time:

$$\mathbf{x}_k = f_k(\mathbf{x}_{k-1}, \mathbf{v}_{k-1}), \tag{2}$$

where f_k is a possible nonlinear function of the state and $\mathbf{v}_k$ is a process noise sequence. The measurement model relates the noisy measurement vector $\mathbf{z}_k$ to the state:

$$\mathbf{z}_k = h_k(\mathbf{x}_k, \mathbf{u}_k), \tag{3}$$

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where h_k is a possible nonlinear function of the state and $\mathbf{u}_k$ is a measurement noise sequence.

The objective of the state space estimation problem is the sequential estimation of the state $\mathbf{x}_k$, $k = 1, 2, \ldots, K$, based on the set of all available measurements up to time k, $\mathbf{z}_i$, $i = 1, \ldots, k$, denoted by $\mathbf{Z}_k$. The state vector of a system can include dynamic system information, such as signal parameters changed by the system or the location and velocity of a target in tracking problems. From a Bayesian perspective, it is required to construct the posterior distribution $p(\mathbf{x}_k|\mathbf{Z}_k)$ to obtain estimates of the states, and it is assumed that the prior distribution $p(\mathbf{x}_0|\mathbf{z}_0) \equiv p(\mathbf{x}_0)$ is available.

If the process or measurement models are nonlinear or $\mathbf{v}_k$ and $\mathbf{u}_k$ are not Gaussian, then suboptimal estimation solutions can be obtained using the extended Kalman filter or the particle filter instead of the Kalman filter algorithm [8].

B. Particle Filtering

The key idea of particle filtering is to represent the required posterior distribution, $p(\mathbf{x}_k|\mathbf{Z}_k)$, by a set of samples or particles $\mathbf{x}_k^i, i = 1, \cdots, N_s$ with associated weights ω_k^i [8]:

$$p(\mathbf{x}_k|\mathbf{Z}_k) \approx \sum_{i=1}^{N_s} \omega_k^i \delta(\mathbf{x}_k - \mathbf{x}_k^i), \quad (4)$$

and to compute estimates based on these samples and weights.

The sequential important sampling (SIS) particle filter is a Monte Carlo method that samples particles from an importance density $q(\cdot)$. The weights can then be expressed as:

$$\omega_k^i \propto \omega_{k-1}^i \frac{p(\mathbf{z}_k|\mathbf{x}_k^i) p(\mathbf{x}_k^i|\mathbf{x}_{k-1}^i)}{q(\mathbf{x}_k^i|\mathbf{x}_{k-1}^i, \mathbf{z}_k)}. \quad (5)$$

Using (4) and (5), one can obtain an estimate of this state as:

$$\hat{\mathbf{x}}_k = \int_{\mathbf{x}_k} \mathbf{x}_k p(\mathbf{x}_k|\mathbf{z}_k) d\mathbf{x}_k = \sum_{i=1}^{N_s} \omega_k^i \mathbf{x}_k^i.$$

Degeneracy is an undesired problem in particle filters where as the time step increases, all but one particle has weights close to zero. When severe degeneracy occurs, resampling is applied to reduce the effect of degeneracy. By resampling, the particles having small weights are eliminated, and new particles are generated from the original particles with larger weights [8].

C. MCMC Methods

MCMC methods including the Gibbs sampling and the Metropolis-Hastings algorithms [9] are a class of algorithms for sampling from probability distributions based on constructing a Markov chain which has the desired distributions.

The Metropolis-Hastings algorithm first generates a candidate point $\mathbf{x}_k^*$ from the jumping distribution $\tilde{q}(\mathbf{x}_{k-1}, \mathbf{x}_k^*)$, which is the probability of returning $\mathbf{x}_k^*$ given the value $\mathbf{x}_{k-1}$ at the last time step. Then the method calculates the ratio of the density, and accepts the candidate with probability:

$$\alpha = \min\left(\frac{\pi(\mathbf{x}_k^*) \tilde{q}(\mathbf{x}_k^*, \mathbf{x}_{k-1})}{\pi(\mathbf{x}_{k-1}) \tilde{q}(\mathbf{x}_{k-1}, \mathbf{x}_k^*)}, 1\right),$$

where $\pi(\mathbf{x})$ is the desired distribution.

D. Static Parameter Estimation

Since particle filtering was developed for dynamic states, it was shown to fail when used to estimate static parameters $\mathbf{x}_k = \mathbf{x}$ [10]. The algorithm described here for the sequential Bayesian estimation of unknown static parameters uses the SMCMC method that combines SIS particle filtering with MCMC methods in [7].

Suppose at time step k, the particles and their corresponding weights, $(\mathbf{x}^1, w_k^1), (\mathbf{x}^2, w_k^2), \cdots, (\mathbf{x}^{N_s}, w_k^{N_s})$, are used to represent $p(\mathbf{x}|\mathbf{Z}_k)$. The SMCMC method updates the weight of the ith particle using:

$$w_{k+m}^i \propto p(\mathbf{z}_{k+1}, \cdots, \mathbf{z}_{k+m}|\mathbf{x}^i, \mathbf{z}_k) w_k^i, \quad (6)$$

where m is the batch size and $p(\mathbf{z}_{k+1}, \cdots, \mathbf{z}_{k+m}|\mathbf{x}^i, \mathbf{z}_k)$ is always available from the measurement model in (3).

Using the Kullback-Leibler distance measure, a rejuvenation test is then performed

$$\kappa(\omega_{k+m}, \omega_k) = \sum_{i=1}^{N_s} \omega_{k+m}^i (\log \omega_{k+m}^i - \log \omega_k^i), \quad (7)$$

to monitor the degeneracy. If degeneracy occurs, the MCMC method with the independent Metropolis-Hastings (IMH) algorithm is used to obtain the proposed density $p(\mathbf{x}|\mathbf{Z}_{k+m})$ by the Gaussian distribution $\mathcal{N}(\mathbf{x}|\mu_\mathbf{x}, \Sigma_\mathbf{x})$ with

$$\mu_\mathbf{x} = \sum_{i=1}^{N_s} w_{k+m}^i \mathbf{x}^i, \quad (8)$$

$$\Sigma_\mathbf{x} = \sum_{i=1}^{N_s} w_{k+m}^i (\mathbf{x}^i - \mu_\mathbf{x})(\mathbf{x}^i - \mu_\mathbf{x})^T. \quad (9)$$

The batch size m is the step distance between two rejuvenations. It varies adaptively depending on the outcome of the rejuvenation test; a large batch means that degeneracy occurs less frequently.

In summary, the SMCMC method takes the following steps:

- At time $(k+1)$, the particles and corresponding weights: $(\mathbf{x}^1, \omega_k^1), (\mathbf{x}^2, \omega_k^2), \cdots, (\mathbf{x}^{N_s}, \omega_k^{N_s})$ are used to represent the distribution $p(\mathbf{x}|\mathbf{Z}_k)$. Set $m = 1$.
- Perform SIS: weights are updated using (6).
- Perform the rejuvenation test:
  If $\kappa(\omega_k, \omega_{k+m}) <$ threshold, then accept the new weights, and $m = m + 1$
  else perform MCMC: sample new particles from $\mathcal{N}(\mathbf{x}|\mu_\mathbf{x}, \Sigma_\mathbf{x})$ where $\mu_\mathbf{x}$ and $\Sigma_\mathbf{x}$ are calculated using (8) and (9), and set $k = k + m$.

When $\mathbf{x}$ can originate from different types of signal structures $\{H_1, H_2, \cdots, H_M\}$, model selection is used simultaneously [7]. Let the parameter vector for model j be $\mathbf{x}^{(j)}$, then, the resulting distribution can be obtained using:

$$p(\mathbf{x}|\mathbf{Z}_k) = \sum_{j=1}^{M} P(H_j|\mathbf{Z}_k) p(\mathbf{x}^{(j)}|\mathbf{Z}_k, H_j),$$

where $P(H_j|\mathbf{Z}_k)$ is the probability of model H_j given measurements $\mathbf{Z}_k$, and $p(\mathbf{x}^{(j)}|\mathbf{Z}_k, H_j)$ is obtained using SMCMC as discussed above given model H_j. Note that SMCMC has to be performed for each model to obtain the particle presentation of $p(\mathbf{x}^{(j)}|\mathbf{Z}_k, H_j)$.

III. Proposed IF Estimation Approach

We propose here a Bayesian IF estimation approach that combines particle filtering and MCMC methods to estimate the static parameters of piecewise linear functions that approximate the IF. Specifically, we approximate the IF $\zeta_s(t)$ as a linear combination of the IFs of non-overlapping linear FM chirps whose FM rates and initial frequencies are unknown. These fixed parameters are sequentially estimated using the SMCMC method discussed in the previous section, and this approach does not require prior knowledge of the signal's phase function $\varphi(t)$.

A. IF Estimation of Single Component Signals

If we define a single component signal as one that can be decomposed into basis functions which are non-overlapping in the TF plane, then its IF is the sum of the IFs of these basis functions. Specifically, if we assume that $s(t)$ can be well-represented by a linear combination of known basis functions:

$$s(t) = \sum_{l=1}^{L} a_l(t) e^{j2\pi\varphi_l(t)} p_l(t), \tag{10}$$

where $p_l(t) = u(t-(l-1)T) - u(t-lT)$ is a rectangular window and $u(t)$ is the unit step, then we can show that the IF of the signal $s(t)$ is given by:

$$\zeta_s(t) = \sum_{l=1}^{L} \frac{d\varphi_l(t)}{dt} p_l(t).$$

As we want to approximate an unknown IF as a piecewise linear function, we let the basis functions in (10) be linear FM chirps with phase $\varphi_l(t) = f0_l t + \frac{1}{2} c_l t^2$ where $f0_l$ is the initial frequency of the lth FM chirp and c_l is its FM rate. Specifically, we approximate $\zeta_s(t)$ by:

$$\zeta_s(t) \approx \sum_{l=1}^{L} (f0_l + c_l t) p_l(t). \tag{11}$$

With this assumption, the estimation of the IF becomes the estimation of a set of unknown static parameter vectors $\mathbf{x}_l = [f0_l, c_l],\ l = 1, \ldots, L$ as shown in Fig. 1 using the SMCMC method described in Section II-D.

In each window l, a linear function is used to approximate the true IF and a new set of particles is used to estimate the parameter vector $\mathbf{x}_l$. The duration of the window has to be carefully chosen to reduce approximation errors. In particular, we consider short duration windows $p_l(t)$ to reduce the error in the piecewise linear approximation. The window length, however, has to be long enough to reduce errors in estimating the static parameters, so no estimation errors occur in the SMCMC. Note that although we assume here that the window has a fixed length, the window can vary in length, as shown in Fig. 1.

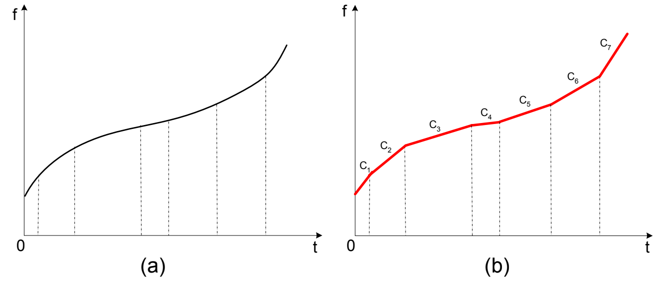


Fig. 1. (a) True IF of a single component signal. (b) Estimated IF using a linear combination of seven piecewise continuous linear FM chirps with indicated FM rates c_l, $l = 1, \ldots, 7$. The vertical lines are used to indicate the starting times of the chirps.

B. IF Estimation of Multiple Component Signals

Whereas we assumed that a single component signal does not have any overlapping segments in the TF plane, we will assume that a multicomponent signal has Q segments that overlap. For example, the signal in Fig. 2 (a) is a two-component signal as it consists of $Q = 2$ distinct non-overlapping curves in the TF plane. As a result, the IF tranform in (1) cannot be used to provide a true representation of a multicomponent signal's frequency variation in time. Here, we propose to use our piecewise linear function approximation to provide an estimate of the sum of the IFs of the different signal components. Specifically, we will consider next two cases where Q is first assumed known and then unknown.

1) Case I: Consider a multicomponent signal with Q components, where Q is known, that can be decomposed in a piecewise linear fashion as:

$$s(t) = \sum_{q=1}^{Q} \sum_{l=1}^{L} a_l^{(q)}(t) e^{j2\pi\varphi_l^{(q)}(t)} p_l(t),$$

where $a_l^{(q)}(t)$ and $\varphi_l^{(q)}(t) = f0_l^{(q)} t + \frac{1}{2} c_l^{(q)} t^2$ are the amplitude and phase of the qth linear FM chirp component in the lth window. Then the IF of this component can be approximated as:

$$\zeta_l^{(q)}(t) \approx f0_l^{(q)} + c_l^{(q)} t, \tag{12}$$

where $\mathbf{x}_l^{(q)} = [f0_l^{(q)}, c_l^{(q)}]$ is the coefficient vector of component q in the lth window. Therefore, in the lth window, Q linear FM chirps are used to approximate the IF and Q parameter vectors, $\mathbf{x}_l^{(q)},\ q = 1, \cdots, Q$, are estimated using SMCMC with a new set of particles for each vector.

2) Case II: When Q is unknown, we assume that there are several models corresponding to the number of components that exist in the lth window. For each model, Q_l linear FM chirps ($Q_l \in [1, 2, \cdots, Q]$) are used to approximate the IFs of the Q_l components in the lth window

$$s(t) = \sum_{q=1}^{Q_l} \sum_{l=1}^{L} a_l^{(q)}(t) e^{j2\pi\varphi_l^{(q)}(t)} p_l(t),$$

with $2Q_l$ parameters $\mathbf{x}_l^{(q)} = [f0_l^{(q)}, c_l^{(q)}],\ q = 1, \cdots, Q_l$ that must be estimated.

We determine the number of components Q_l using model selection. Assuming that Q_l could vary in each window but that the maximum number is known to be Q, then there are a total of $Q+1$ models, including the possibility of no signal:

$$\begin{aligned} H_0: \quad & z_l(t) &= u_l(t) \\ H_1: \quad & z_l(t) &= a_l(t)e^{j2\pi\varphi_l(t)} + u_l(t) \\ & & \vdots \\ H_Q: \quad & z_l(t) &= \textstyle\sum_{q=1}^{Q} a_l^{(q)}(t)e^{j2\pi\varphi_l^{(q)}(t)} + u_l(t). \end{aligned}$$

The Q_l signal component parameter vectors in the lth window are estimated simultaneously. For example, in Fig. 2, the signal IF is approximated by one (model H_1) or two (model H_2) linear FM chirps in each window.

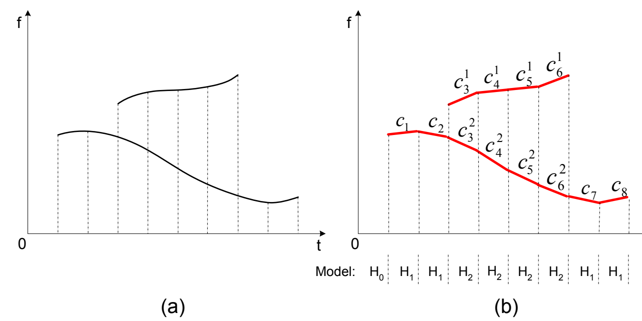


Fig. 2. (a) True IFs of a multiple component signal with the number of components ($Q = 2$) unknown. (b) Estimated IFs using model selection and a linear combination of non-overlapping linear FM chirps with indicated FM rates $c_l^{(q)}$.

IV. Simulations

A. Single Component Signals

1) Known IF Structure: If the IF structure of the signal is known to be linear, we only need to find c and $f0$ following [7]. In this simulation, a 3 dB noisy linear FM chirp was used with the FM rate $c = 6$ Hz^2 and initial frequency $f_0 = 20$ Hz. Fig. 3 shows the estimation results of the parameters. As it can be seen, the estimates approach the true values after only a few time steps.

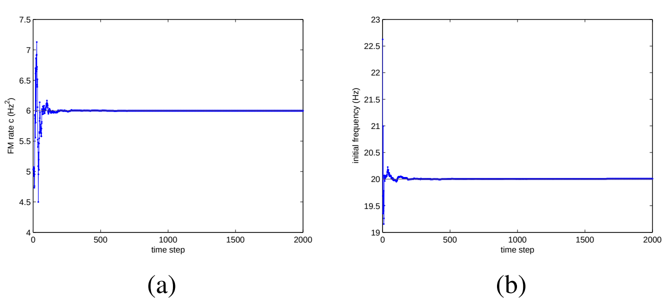


Fig. 3. Estimated values of (a) FM rate, and (b) initial frequency.

2) Simultaneous Model Selection and Parameter Estimation: We simulated the simultaneous model selection and corresponding parameter estimation of a single component signal. Assume the signal was chosen from one of four possible models,

$$\begin{array}{lll} H_0: & \text{noise only,} & z(t) = u(t) \\ H_1: & \text{complex sinusoid,} & z(t) = Ae^{j2\pi f_0 t} + u(t) \\ H_2: & \text{linear FM chirp,} & z(t) = Ae^{j2\pi ct^2} + u(t) \\ H_3: & \text{hyperbolic FM chirp,} & z(t) = Ae^{j2\pi c \ln t} + u(t), \end{array} \quad (13)$$

where $u(t)$ is zero-mean additive white Gaussian noise. Our measurements $z(t)$ were generated from a linear FM chirp with FM rate $c = 3$ Hz^2 and 3 dB SNR. Fig. 4 (a) demonstrates the estimated probabilities of the four models with respect to time, beginning with equal probability of $1/4$, and Fig. 4 (b) shows the estimated parameter value. The correct model H_2 (chirp) is selected, and the FM rate is estimated correctly.

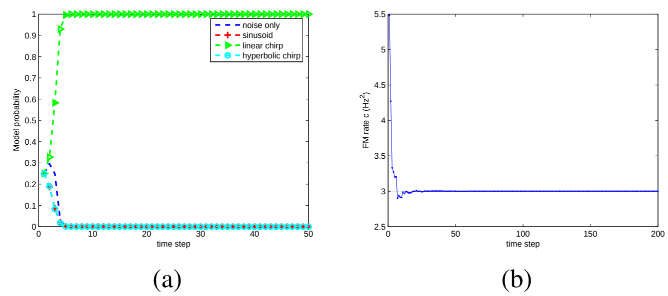
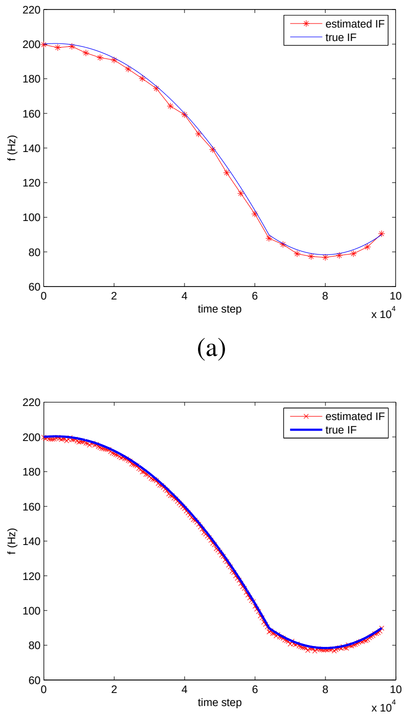


Fig. 4. (a) Probability of each model in (13) (only the first 50 samples are shown). (b) Estimated value of FM rate.

3) Unknown IF Structure: If the IF structure is unknown, piecewise linear approximation is applied as described in Section III-A. We demonstrate the estimation performance by comparing the true and estimated IF of a 9 dB SNR noisy single component signal in Fig. 5 with window length (a) 4000 samples and (b) 500 samples. As it can be seen, a shorter window length yields better performance results. Note that by time-averaged root mean squared error (RMSE) in Fig. 6, we mean the RMSE avaraged over all time steps and Monte Carlo iterations. Fig. 6 (a) demonstrates the dependence of the estimation performance on the fixed length T of the rectangular window $p_l(t)$ in (10). We note that, while there is no dramatic difference in computational complexity, the shorter the window length, the better the performance, provided that the window is long enough to reduce errors in estimating the linear FM chirp parameters using the SMCMC. Fig. 6 (b) shows the effect of different values of SNR on the estimation performance.

B. Multiple Component Signals

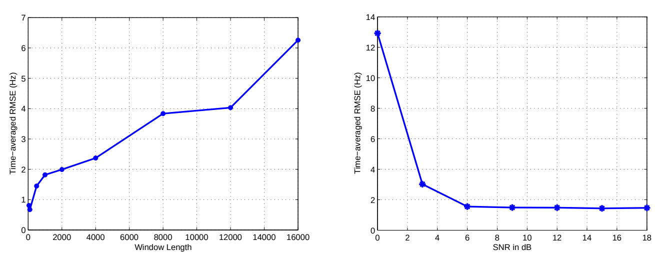
Next, we demonstrate the use of SMCMC with model selection for multicomponent signal IF estimation. In Fig. 7, the number of components was known to be $Q = 2$. After the estimation, two linear FM chirps (each with duration $96,000$ time steps) were obtained that completely overlapped in time.



(a)

(b)

Fig. 5. IF estimation compared with true IF using piecewise linear approximation with window length: (a) $T = 4000$ samples, (b) $T = 500$ samples in (10).



(a) (b)

Fig. 6. Time-averaged RMSE for: (a) different window lengths, (b) different SNR values.

In Fig. 8, the two signal components have different TF structures, and we assumed that the number of signal components was unknown. Note that when the two components overlap in time, the estimation of the lower frequency signal component is not as accurate as in Fig. 7 because of the nonlinearity of the lower frequency component in Fig. 8.

V. Conclusion

We proposed an IF estimation technique that uses particle filtering and MCMC methods to estimate static parameters of piecewise linear functions in a sequential approach. This allows for the rapid adaption to changing signal characteristics, and thus for the IF characterization of both single and multiple

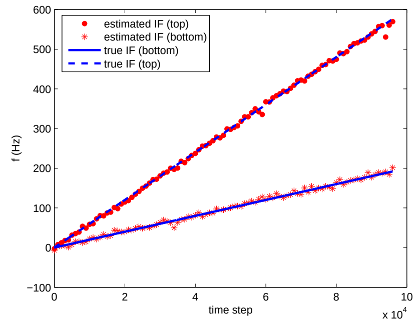


Fig. 7. IF estimation of a signal consisting of two overlapping linear FM chirps with the number of components known.

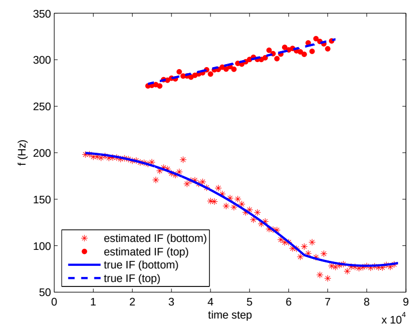


Fig. 8. IF estimation of a signal consisting of two components.

component signals. We are currently comparing our approach with other existing IF estimation techniques.

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