Golay Complementary Waveforms for Sparse Delay-Doppler Radar Imaging

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***Abstract*—We present a new approach to radar imaging that exploits *sparsity in the matched filter domain* to enable high resolution imaging of targets in delay and Doppler. We show that the vector of radar cross-ambiguity values at any fixed test delay cell has a sparse representation in a Vandermonde frame that is obtained by discretizing the Doppler axis. The expansion coefficients are given by the auto-correlation functions of the transmitted waveforms. We show that the orthogonal matching pursuit (OMP) algorithm can then be easily used to identify the locations of the radar targets in delay and Doppler. Unambiguous imaging in delay is enabled by alternating between a Golay pair of phase coded waveforms at the transmission to eliminate delay sidelobe effects. We then extend our work to multi-channel radar, by developing a sparse recovery approach for dually-polarimetric radar. We exploit sparsity in a bank of matched filters, each of which is matched to an entry of an Alamouti matrix of Golay waveforms to recover a co-polar or cross-polar polarization scattering component.**

***Index Terms*—Golay complementary waveforms, sparse representations, Vandermonde frame, orthogonal matching pursuit, delay-Doppler imaging.**

I. INTRODUCTION

The advent of sparse representation theory has potentially pervasive impact across the spectrum of sensing applications. One of the applications that has been receiving increasing attention is active radar imaging, where the goal is to resolve a small number of targets that sparsely populate the radar field.

Baranuik and Steeghs [1] were the first to exploit sparse representations for radar detection. Their motivation was to improve resolution limits imposed on radar imaging due to technological and economical constraints associated with the use of high-rate analog-to-digital (A/D) converters for processing ultra-wideband (high resolution) radar pulses. Shortly after, Herman and Strohmer [2] proposed a formal mathematical framework, based on compressed sensing (CS) theory, for reconstructing sparse scattering fields. The essential idea in [2] is that the return from a radar scene will generally be a sparse superposition of time-shifted and (under narrowband approximation assumptions) frequency-shifted replicates of the transmitted waveform. The CS theory (cf. [3],[4]) thus suggests that a few projections onto measurement functions incoherent with this sparsity set should be sufficient to identify which elements of the sparsity set are present in the return, thereby providing a high-resolution analysis of the scene.

One issue with the approach in [2] is that the optimal detector structure built into most radar receivers, i.e. a matched filter, is absent in the proposed receiver and consequently the results are only favorable in high SNR regimes. From classical detection theory, detection of a known signal (in this case, the transmitted signal) in additive Gaussian noise is optimized by matched filtering. The cross-ambiguity function between the transmitted and received signals effectively represents an array of matched filter outputs corresponding to time shifts and frequency shifts of the transmitted signal, hence encodes the information necessary for optimal detection. Further, the cross ambiguity function itself is typically a noisy sparse superposition of terms defined by the auto-ambiguity function of the transmitted waveforms.

In this paper, we present a new approach to radar imaging that exploits *sparsity in the matched filter domain.* We transmit a pulse train of phase coded waveforms and cross-correlate the radar returns with their corresponding transmit waveforms at each pulse repetition interval. We show that the vector of cross-correlation values at a given lag (test delay cell) has a sparse representation in a Vandermonde frame, whose elements are determined by the discretization of the Doppler axis. The expansion coefficients are given by the autocorrelation values of the transmitted waveforms at the delay cell of interest. The orthogonal matching pursuit (OMP) algorithm (cf. [5]) is then used to identify the location of the radar targets in delay and Doppler. We note that any single phase coded waveform has an aperiodic autocorrelation function with nonzero sidelobes. These sidelobes prohibit unambiguous imaging of two targets that have nonzero coefficients on the same Vandermonde frame vector. We show that by alternating between a pair of Golay complementary waveforms (cf. [6],[7]) at the transmission, delay sidelobe effects can be avoided and targets can be resolved in delay and Doppler. We then extend our work to multi-channel radar, by developing a sparse radar imaging approach for dually-polarimetric radar. We combine Alamouti waveform matrices of Golay pairs, originally developed in [8],[9], with sparse representations in Vandermonde frames over each polarization channel to resolve targets based on their full polarization scattering properties. This amounts to applying OMP for sparse target recovery from the outputs of a unitary bank of matched filters, each of which is matched to a Golay complementary waveform.

A fundamental assumption in our approach is that the discretization of the Doppler axis, which produces the Vandermonde frame for sparse representation, is fine enough that

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the targets can be assumed to fall almost exactly on the grid. The analysis of the sensitivity of our approach to mismatch between the Vandermonde frame assumed for sparse radar imaging and the actual basis (corresponding to actual Doppler points) is beyond the scope of this paper. However, a recent study [10] shows that sparse recovery algorithms based on basis pursuit in CS are relatively sensitive to mismatch in the sparsifying basis. This study shows that CS algorithms may not be reliable for modal analysis (which is the primary objective in radar and sonar) in the presence of basis mismatch. The reader is referred to [10].

Remark: We note that several CS approaches to passive radar exist. The reader is referred to [11] and the references therein for a list of relevant literature.

II. Sparse representation of a radar scene

A. Radar returns

Consider a baseband radar waveform $s_x(t)$ phase coded by a length-L unimodular sequence x, i.e.,

$$s_x(t) = \sum_{\ell=0}^{L-1} x(\ell)\Omega(t - \ell T_c), \tag{1}$$

where $\Omega(t)$ is a unit energy pulse shape supported mostly on $(0, T_c)$ and T_c is the chip length.

Assume $s_x(t)$ is transmitted M times, with a pulse repetition interval (PRI) of T seconds between consecutive transmissions. Suppose there're N_t targets parameterized by the delay-Doppler pairs $\{(\tau_i, \nu_i)\}_{i=1}^{N_t}$ in the radar scene, in the mth PRI the transmitted signal $s_m(t)$ reflects back from each target after a round-trip delay τ_i and Doppler shift ν_i. The received signal after demodulation and low-pass filtering is given as a superposition of returns from all targets:

$$r_m(t) = \sum_{i=0}^{N_t-1} A_i s_m(t - mT - \tau_i) e^{j\nu_i(t-mT)} e^{j\phi_i}, \tag{2}$$

where A_i's are target scattering coefficients and $e^{j\phi_i}$'s are carrier phase shifts due to propagation delays τ_i's.

The matched filter output for the mth PRI, sampled in delay at a of rate $1/T_c$, is given by

$$w_m(k) = \sum_{i=0}^{N_t-1} h_i D_i^m C_x(k - d_i),\ 0 \le m \le M-1, \tag{3}$$

where $h_i = A_i e^{j\phi_i} e^{j\nu_i \tau_i}$, $d_i = \lceil \tau_i / T_c \rceil$ is the discretized delay and $D_i = e^{j\nu_i T}$ is the Doppler of the targets respectively. Here we have only considered relative Doppler shifts over PRIs and have dropped the negligible Doppler shifts over the duration of $s_x(t)$.

B. Sparse representation in a Vandermonde frame for Doppler imaging

Assume that the Doppler axis is finely discretized to N cells represented by $\mathcal{D} = \{\omega_n\}_{n=0}^{N-1}$, where $\omega_n = e^{j2\pi n/N}$. We consider the idealized situation where the Doppler shift of all targets, over consecutive PRIs, are located in $\mathcal{D}$.

Definition 1: We define a *Vandermonde frame* $\mathcal{F}$ as the set of rows of the matrix $\mathbf{V}^T$ given by

$$\mathbf{V}^T = \begin{bmatrix} 1 & \omega_0 & \omega_0^2 & \cdots & \omega_0^{M-1} \\ 1 & \omega_1 & \omega_1^2 & \cdots & \omega_1^{M-1} \\ 1 & \omega_2 & \omega_2^2 & \cdots & \omega_2^{M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_{N-1} & \omega_{N-1}^2 & \cdots & \omega_{N-1}^{M-1} \end{bmatrix}, \tag{4}$$

where M is the total number of PRIs in transmission. Thus, $\mathcal{F} = \{\eta_n = (1, \omega_n, \omega_n^2, \cdots, \omega_n^{M-1})\}_{n=0}^{N-1}$.

Let $\mathbf{w}(k)$ be the vector of all matched filter outputs $w_m(k)$, $m = 0, \ldots, M-1$ at a fixed delay cell k, that is

$$\mathbf{w}(k) = [w_0(k), w_1(k), \cdots, w_{M-1}(k)]^T. \tag{5}$$

We claim $\mathbf{w}(k)$ has a sparse representation in the Vandermonde frame $\mathcal{F}$.

Theorem 1: The matched filter output vector $\mathbf{w}(k)$ is an N_t-sparse vector in the Vandermonde frame $\mathcal{F}$, that is, $\mathbf{w}(k) = \mathbf{V}\mathbf{x}$ where $\mathbf{x}$ is N_t-sparse, if and only if all target Doppler shifts $D_i \in \mathcal{D}$. This representation is unique if $N_t \le M/2$.

Proof: It is enough to show that when there's only one target located on the delay-Doppler plane at (d_i, D_i) and $D_i \in \mathcal{D}$, $\mathbf{w}(k)$ is a one-sparse vector corresponding to $\eta_i = (1, D_i, D_i^2, \cdots, D_i^{M-1})$. Write $\mathbf{w}(k)$ as

$$\mathbf{w}(k) = \begin{bmatrix} C_x(k-d_i) \\ D_i C_x(k-d_i) \\ \vdots \\ D_i^{M-1} C_x(k-d_i) \end{bmatrix} = C_x(k-d_i)\eta_i^T, \tag{6}$$

$\mathbf{w}(k)$ is mapped to η_i^T, and the corresponding coefficient is the autocorrelation of x delayed by d_i at the kth delay cell.

The unique representation part in Theorem 1 is essentially the same as Lemma 2.2 in [12]. ■

Remark: The condition $N_t \le M/2$ means the number of PRIs transmitted should be twice greater than the number of targets, which is true in most radar applications.

Theorem 1 shows that the problem of resolving radar targets in Doppler across a fixed delay cell can be reduced to a sparse recovery problem, where the goal is to recover a sparse vector given the observation $\mathbf{w}(k) = \mathbf{V}\mathbf{x}$ for all k, possibly with additive noise to $\mathbf{w}(k)$. In this paper, we use orthogonal matching pursuit (OMP) as our sparse recovery algorithm. The OMP is simple to implement and numerical evidence suggests that as long as the Vandermonde frame is sufficiently incoherent the OMP algorithm is efficient for signal recovery. The reader is referred to [5] for details.

C. Delay sidelobes

Theorem 1 enables the use of sparse representations for Doppler imaging across a fixed delay line. To resolve targets in both delay and Doppler the OMP needs to be applied to matched filter output vectors $\mathbf{w}(k)$ for all delay k. If the autocorrelation function $C_x(k)$ were a delta function then this procedure would identify exactly N_t targets at the correct delay and Doppler locations. However, any single phase coded waveform has an aperiodic autocorrelation function with nonzero sidelobes in delay. These sidelobes prohibit unambiguous imaging of targets that have the same Doppler coordinates, as sidelobes of stronger targets might mask the weak targets.

To highlight this problem, we look at a simple radar scenario with five point targets. The parameters of the targets are $(1000, 0, 0)$, $(1200, 0, -20)$, $(1500, -40\pi/N, 0)$, $(1500, 12\pi/N, -20)$ and $(1800, 53\pi/N, 0)$, where $N = 256$ is the number of cells on the Doppler axis. The first parameter is the distance between the radar and targets in meters, the second is the Doppler of targets in rad, and the third is the amplitudes of targets in dB. The SNR is 10dB with respect to the strongest target. The same phase coded waveform is transmitted over 64 PRIs. The delay-Doppler image in Figure 1(a) shows the result of sparse reconstruction in the Vandermonde frame $\mathcal{F}$ across all delay cells. We notice that the weaker targets are masked by the delay sidelobes of the stronger targets.

III. Golay complementary waveforms for sidelobe suppression

We now show that delay sidelobes can be pushed away from an imaging region of interest by alternating between a pair of Golay complementary waveforms across PRIs.

Definition 2: Two length L unimodular sequences of complex numbers x_1 and x_2 are Golay complementary if for $k = -(L-1), \ldots, (L-1)$ the sum of their autocorrelation functions satisfies

$$C_{x_1}(k) + C_{x_2}(k) = 2L\delta(k), \tag{7}$$

where $C_{x_1}(k)$ is the autocorrelation of x_1 at lag k and $\delta(k)$ is the Kronecker delta function. Each member of the pair (x_1, x_2) is called a Golay sequence.

The baseband waveforms $s_{x_1}(t)$ and $s_{x_2}(t)$ are phased-coded by x_1 and x_2 respectively. Our approach is to transmit Golay complementary waveforms $s_{x_1}(t)$ and $s_{x_2}(t)$ alternatively over M PRIs. At the mth PRI, we transmit $s_m(t) = s_{x_1}(t)$ if m is even and $s_m(t) = s_{x_2}(t)$ is m is odd.

One way to interpret the Golay complementary property is that $C_{x_1}(0) = C_{x_2}(0) = L$ and $C_{x_1}(k) = -C_{x_2}(k)$ for $k \neq 0$. This makes $\mathbf{w}(k)$ select different vectors in the Vandermonde frame depending on if k equals the target delay d_i. Assume there exists a target (d_i, D_i) in the radar scene, when $k = d_i$,

$$\mathbf{w}(d_i) = \begin{bmatrix} C_{x_1}(0) \\ D_i C_{x_2}(0) \\ \vdots \\ D_i^{M-1} C_{x_2}(0) \end{bmatrix} = L\eta_i^T. \tag{8}$$

When $k \neq d_i$,

$$\mathbf{w}(k) = \begin{bmatrix} C_{x_1}(k-d_i) \\ D_i C_{x_2}(k-d_i) \\ \vdots \\ D_i^{M-1} C_{x_2}(k-d_i) \end{bmatrix} = C_{x_1}(k-d_i)\eta_{i+N/2}^T, \tag{9}$$

where $\eta_{i+N/2} = (1, -D_i, D_i^2, \cdots, -D_i^{M-1})$. This corresponds to a Doppler shift of π.

If all target Dopplers are within the interval $[-\pi/2, \pi/2]$, the sidelobes from the sparse recovery will be pushed to the Doppler intervals $[-\pi, -\pi/2]$ and $[\pi/2, \pi]$. This allows us to identify all targets without ambiguity and remove sidelobes inside the Doppler interval $[-\pi/2, \pi/2]$. The following theorem summarizes this result.

Theorem 2: If Golay complementary waveforms are transmitted in an alternating sequence, then the sparse reconstruction from the Vandermonde frame $\mathcal{F}$, as outlined below, will unambiguously recover the delay and Doppler coordinates of all $N_t \leq M/2$ targets, provided that all Doppler coordinates are between $[-\pi/2, \pi/2]$ and they belong to $\mathcal{D}$.

Algorithm 1 Sparse Recovery Algorithm

```
for all i such that 0 ≤ i ≤ M − 1 do
    if mod(i, 2) = 0 then
        w_i(k) ⇐ match filtering with the Golay sequence x_1;
    else
        w_i(k) ⇐ match filtering with the Golay sequence x_2;
    end if
end for
Let P be the length of the match filtered sequence.
for all k such that 0 ≤ k ≤ P − 1 do
    Let w(k) = [w_0(k), w_1(k), ··· , w_{M−1}(k)]^T
    y(k) ⇐ OMP of w(k) in the Vandermonde frame F.
end for
Form an image of [|y(0)|, |y(1)|, ···, |y(P − 1)|].
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Let us consider the same scenario as in Figure 1(a), but this time we alternate between a pair of Golay complementary waveforms at transmission. Figures 1(b),(c) show the delay-Doppler image obtained using sparse recovery for this case. We observe that the alternating sequence of Golay complementary waveforms, in conjunction with sparse reconstruction in Vandermonde frames, enables perfect recovery of targets in delay and Doppler inside the $[-\pi/2, \pi/2]$ Doppler band.

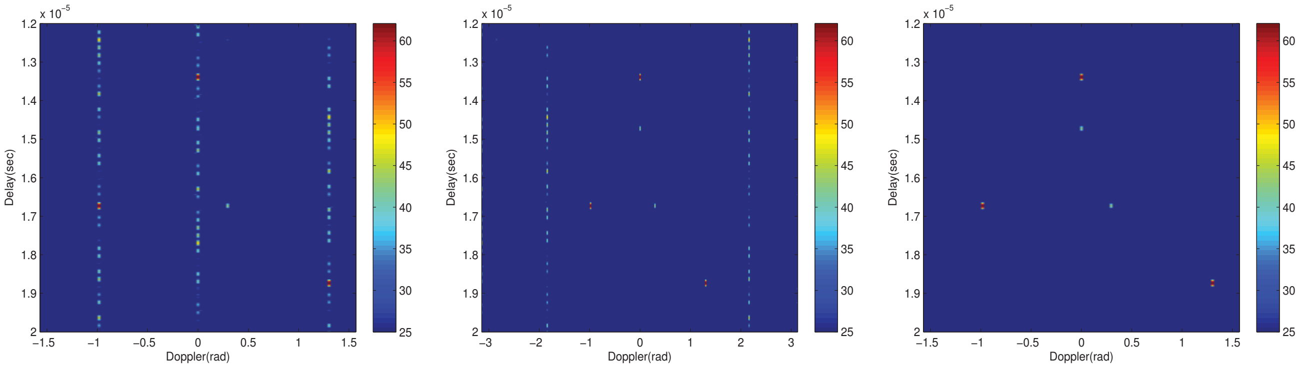
IV. Generalizations to higher dimension

In [8],[9], a simple scheme of transmitting Golay complementary waveforms in an Alamouti fashion is utilized to enable instantaneous radar polarimetry. Here we integrate this Alamouti scheme with our sparse representation approach to enable recovery of targets over orthogonal polarization channels and to resolve targets based on their full polarization scattering properties.

Definition 3: Let (x_1, x_2) be a Golay complementary pair. The vector-valued sequences $\mathbf{u}$ and $\mathbf{v}$ are called *vector-valued Golay complementary* if they satisfy

$$\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} -\tilde{x}_2 \\ \tilde{x}_1 \end{bmatrix}. \tag{10}$$

where $\tilde{x}_1[\ell] = x_1^*[L-1-\ell]$ and $\tilde{x}_2[\ell] = x_2^*[L-1-\ell]$ for $0 \leq \ell \leq L-1$.



(a) Single waveform; Doppler band $[-\pi/2, \pi/2]$ rad

(b) Alternating Golay waveforms; The entire Doppler band

(c) Alternating Golay waveforms; Doppler band $[-\pi/2, \pi/2]$ rad

Fig. 1. Sparse reconstruction from a single waveform and alternating Golay complementary waveform transmissions; $N = 256$ and $M = 64$.

Since the Golay property is invariant to time reverse and change in signs, $(-\tilde{x}_2, \tilde{x}_1)$ is also a Golay complementary pair. The vector-valued Golay pair $(\mathbf{u}, \mathbf{v})$ satisfies the property that the sum of their complex-valued auto-correlation matrices is proportional to identity:

$$\mathbf{C}_{\mathbf{u}}(k) + \mathbf{C}_{\mathbf{v}}(k) = \begin{bmatrix} 2L\delta(k) & 0 \\ 0 & 2L\delta(k) \end{bmatrix}. \quad (11)$$

We transmit $(\mathbf{u}, \mathbf{v})$ alternatively in a dually-polarized radar system. It is possible to separate the four polarized channels using linear processing at the receiver end. The return waveforms from two polarizations are processed through four matched filters: the return $\mathbf{r}_m$ in the mth PRI is matched filtered with phase-coded waveform $\mathbf{u}$ when m is even; and matched filtered with phase-coded waveform $\mathbf{v}$ when m is odd. The output of the matched filter can be written in a 2×2 matrix form $\mathbf{W}_m(k)$ for $0 \leq m \leq M - 1$. Similarly, we form a matrix array as

$$\mathbb{W}(k) = [\mathbf{W}_0(k), \mathbf{W}_1(k), \cdots, \mathbf{W}_{M-1}(k)]^T. \quad (12)$$

It is easy to show by the same arguments in Theorem 1, the vector formed from each of the matched filter outputs $[\mathbb{W}(k)]_{ij}$ for $i, j = 1, 2$ will have a sparse representation in the Vandermonde frame. The unambiguous imaging in delay and Doppler is obtained by performing orthogonal matching pursuit for all four matched filter outputs in parallel.

V. Conclusions

We present a new approach to radar imaging that exploits sparsity in the matched filter domain to efficiently resolve targets in delay and Doppler. This is enabled by the observation that the radar cross-ambiguity vector, indexed by PRIs, across a fixed test delay cell has a sparse representation in a Vandermonde frame. The base elements in the Vandermonde frame are obtained by discretizing the Doppler axis. An orthogonal matching pursuit algorithm can then be used to resolve targets in Doppler by recovering the sparse coefficient vector, whose nonzero entries are given by the autocorrelation function of the transmit waveform at the delay cell of interest. To avoid returning replicas of the same target across a fixed Doppler line, due to the presence of delay sidelobes in the autocorrelation function of the transmit waveform, we transmit a pair of Golay complementary waveforms in an alternating fashion. The interplay between this alternating pattern and the complementary property of Golay sequences enables pushing of delay sidelobes to the outside of the imaging region of interest. An extension of the sparse imaging concept for dually-polarimetric radar is also presented. This extension combines our previous work on instantaneous radar polarimetry using Alamouti matrices of Golay waveforms with sparse recovery in Vandermonde frames.

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