Information Theoretic Radar Waveform Design for Multiple Targets

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***Abstract*— In this paper we use information theoretic approach to design radar waveforms suitable for simultaneously estimating and tracking parameters of multiple targets. Our approach generalizes the information theoretic water-filling approach of Bell. The paper has three main contributions: A new information theoretic design criteria for single transmit waveform with a receiving array using a weighted linear sum of the mutual informations between targets' radar signatures and the corresponding received beams (given the transmitted waveforms), we proivde a family of design criteria that weight the various targets according to priorities. Then we generalize the information theoretic design criteria for designing multiple waveforms under joint power constraint when beamforming is used both at transmitter and receiver. Finally we provide a highly efficient optimization algorithm for optimizing the transmitted waveforms both for single target and multiple targets. We show that the optimization problem in both cases can be decoupled into a parallel set of low dimensional search problems at each frequency, with dimension defined by the number of targets, instead of the number of frequency bands used. The power constraint is forced through the optimization of a single Lagrange multiplier for the dual problem. We end with comments on the generalization of the proposed technique for other design criteria, e.g., for the linearly weighted MMSE design criterion.**

I. Introduction

The problem of radar waveform design is of fundamental importance in designing state of the art radar systems. The possibility to vary the transmitted signal on a pulse by pulse basis opens the door to great enhancement in estimation and detection capability as well as improved robustness to jamming. Furthermore modern radars have the possibility to detect and track multiple targets simultaneously. Therefore designing the transmitted pulses for estimating multiple targets becomes a critical issue in radar waveform design.

Most of existing waveform design literature deals with design for a single target. One of the important tools in such design is the use of information theoretic techniques. The pioneering work of Woodward and Davies [1], [2], [3], [4] were the first to suggest that information theoretic tools are important for development of radar receivers. Grettenberg [5] proposed using information theoretic criteria for optimizing radar sensitivity to estimated parameters by formulating the estimation problem as a multiple hypothesis testing at a given parameter space resolution. He proposed to maximize the minimum diveregence between any two hypotheses tested. This reduced to minimizing a largest value of the ambiguity function at a given distance from the origin. Schweppe and Gray [6] proposed design criteria for radar signals under both average and peak power constrains. In [7] that radar signal is designed to optimize the probability of detection of a point target in a clutter. The authors of [7] are optimizing the radar signal over a family of uniformly spaced pulse trains with complex gains to each pulse. Bell [8] has been the first to propose using the mutual information between a random extended target and the received signal. His optimization led to a water-filling type strategy. In his paper he assumes that the radar signature is a realization of random Gaussian process with a known power spectral density (PSD). However when considering real-time signal design his approach can be used to enhance the next transmitted waveform based on the a-priori known signature. It is interesting to note that Bell's formulation is equivalent to the design of the best communication channel intended to deliver a specific Gaussian random signal (under power constraint on channel response).

Whereas waveform design literature concentrated on estimation of a single target, this completely contrasts the capabilities of modern radars to treat multiple targets. Therefore development of design techniques for multiple targets are of critical importance to modern radar waveform design.

Recent advances in convex optimization, see [9] and the references therein, open the way to design techniques specifically tailored for designing of radar waveforms suitable for estimating the parameters of multiple targets.

In this paper we study the problem of radar waveform design for multiple target estimation and tracking based on information theoretic concepts. The paper has three main contributions: First we extend Bell's results to the design of a single wavform for simultaneous estimation and tracking of multiple targets using phased array techniques at the receiver. This approach is then generalized to the case of multiple transmit waveforms, when the transmitter employs beamforming as well. Finally an optimization algorithm is proposed for both cases. For both single and multiple waveform design we show that using duality theory the problem can be reduced to a search over a single parameter and multiple low-dimensional optimization problems at each frequency. Interestingly even though the proposed design criteria for multiple waveforms is non-convex strong duality [9] still holds, which allows us to solve the simpler dual problem. Finally we comment that the same observation enables optimization of a weighted linear sum of the non-causal mean square error.

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II. Information theoretic approach to waveform design

In this section we extend the waveform design paradigm of Bell [8] to the case of multiple radar transmitters and receivers. The section is divided into three parts: After a brief review of the result of [8] we analyze the case of single waveform design for spatially resolved targets. This is interesting when transmitter is simple e.g., in bi-static radar situations. We end up with generalization of our approach to the case of multiple transmit wavefoms, each optimized for a specific target. In order to study the trade off between various radar receivers we use a linear convex combination of the mutual information between the targets and the received signal at each receiver beam oriented at that specific target.

A. The design of a single waveform for a single target

We begin with a brief overview of Bell's information theoretic approach to the waveform design problem. In this paper we limit ourselves to the case of estimation waveforms for extended targets as described in [8]. We assume that the targets are acting on the transmitted waveform as a random linear time invariant system with discrete time frequency response taken from a Gaussian ensemble with known PSD. Denote by $\mathbf{h}(f) = [h(f_1), ..., h(f_K)]^T$ the target's radar signature and by $\sigma_h^2(f_k)$ its PSD at frequency f_k. As noted in [8] extension for the delay-doppler case is possible, but complicates the formulation. A realization of the received signal is given by

$$x(f_k) = h(f_k)s(f_k) + w(f_k) \quad k = 1, \dots, K \tag{1}$$

where $s(f_k)$, $w(f_k)$ are respectively the discrete time waveform and clutter at frequency f_k and K is the number of frequency sub-bands. Under our assumptions and assuming complex envelope signaling over sufficiently narrowband division of the transmit bandwidth the mutual information between the target frequency response and the received signal at frequency f_k is given by

$$I\left(h(f_k); x(f_k)|s(f_k)\right) = \Delta f \log\left(1 + \frac{\sigma_h^2(f_k)|s(f_k)|^2}{\sigma_w^2(f_k)}\right) \tag{2}$$

where $\sigma_w^2(f_k)$ is the clutter PSD at frequency f_k, and Δf is the bandwidth used. The total mutual information between target frequency response and received signal is now given by

$$I\left(\mathbf{h}; \mathbf{x}|\mathbf{s}\right) = \Delta f \sum_{k=1}^{K} \log\left(1 + \frac{\sigma_h^2(f_k)|s(f_k)|^2}{\sigma_w^2(f_k)}\right). \tag{3}$$

To maximize the mutual information Bell proves [8] that a water-filling strategy is required where the transmit PSD is given by

$$|s(f_k)|^2 = max\left\{0, A - \frac{\sigma_w^2(f_k)}{\sigma_h^2(f_k)}\right\} \tag{4}$$

and A is a constant chosen so that the total power constraint is met. It is interesting to note that unlike the usual communication problem the waveform design is similar to optimization of a communication channel for a given signal family rather than optimization of the signal to achieve capacity.

Note that since the target signature is the desired Gaussian signal we have no limitation on the distribution of $s(f_k)$, and the phase can be chosen arbitrarily. This means that we can use almost constant amplitude, by proper frequency scanning using a linear sum of properly delayed and windowed complex exponential with durations proportional to the amplitude $|s(f_k)|^2$.

B. Designing a single transmit waveform for multiple spatially resoved targets

We now turn to the case of multiple targets. We use similar approach to [8] where we use the mutual information as the basis for the waveform design. Moreover similarly to the notion of rate region of the broadcast channel that has been solved recently [10], [11], [12], [13] we look at the waveform design problem as a broadcast channel design problem, where the signaling is given and we are free to choose our optimal channel under total power constraint. We assume a single transmit waveform and multiple receive elements that are used for reception of the multiplle targets. Following [8] we assume that L many targets are taken from a Gaussian ensemble with an a-priori known power spectral densities. In this paper we assume that L is known. However in an extended version of this work we will provide details on the adaptive estimation of L. The PSD of the ℓ'th target at frequency f_k is given by $\sigma_{h_\ell}(f_k)$. We also assume a total power constraint on the transmitted signal, i.e.,

$$P_s = \sum_{k=1}^{K} |s(f_k)|^2 \Delta f \tag{5}$$

The received signal for the ℓ'th beam at frequency f_k in complex envelop form can be described as

$$z_\ell(f_k) = \mathbf{w}_\ell^*(f_k)\left(\sum_{i=1}^{L} \mathbf{a}(\theta_i, f_k)h_i(f_k)\right)s(f_k) + \boldsymbol{\nu}(f_k) \tag{6}$$

where $\mathbf{a}(\theta_i, f_k)$ and $h_i(f_k)$ are the array response towards the direction and frequency response of the i'th target at frequency f_k respectively, and $\mathbf{w}_\ell^*(f_k)$ is the beamformer vector of the ℓ'th beam at frequency f_k. The analysis described can be applied to any beamforming techniques underlying the radar operation, e.g., zero forcing ,MVDR, SMI, LCMV, GSC or derivative constrained beamforming [14]. However we assume that all received beams are known to the radar processing unit. Since the transmitted waveforms are deterministic and the target response is assumed Gaussian, we obtain that the mutual information between the received signal and the ℓ'th target radar signature is given by:

$$\begin{aligned} &I\left(h_\ell(f_k); z_\ell(f_k)|s(f_k)\right) = \\ &\log\left(1 + \frac{\sigma_{h_\ell}^2(f_k)|g_{\ell,\ell}(f_k)|^2|s(f_k)|^2\Delta f}{|g_{\ell,j}(f_k)|^2|s(f_k)|^2\Delta f + \sigma_{\nu_\ell'}^2(f_k)}\right)\Delta f \end{aligned} \tag{7}$$

where we define

$$g_{\ell,i}(f_k) = \mathbf{w}_\ell^*(f_k)\mathbf{a}(\theta_i, f_k) \tag{8}$$

and

$$\sigma_{\nu_\ell'}^2(f_k) = |\mathbf{w}_\ell^*\boldsymbol{\nu}|^2\Delta f \tag{9}$$

to be the complex beam gain of the ℓ'th beam towards the i'th target. Since we assume that the targets are spatially resolved by a linear receive beamformer, all the information for a single target is captured by $\mathbf{z} = [z_\ell(f_1), ..., z_\ell(f_K)]^T$

Integrating over all frequencies we obtain that for each target the mutual information of the target and the received beam is given by:

$$I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{p}\right) = \Delta f \sum_{k=1}^{K} \log\left(1 + \frac{\sigma^2_{h_\ell}(f_k) p_k |g_{\ell,\ell}(f_k)|^2}{\sum_{j\neq\ell} \sigma^2_{h_j}(f_k) p_k |g_{\ell,j}(f_k)|^2 + \sigma^2_{\nu'_\ell}(f_k)}\right) \tag{10}$$

where $\mathbf{p} = [p_1, ..., p_K]$ and $p_k = p(f_k) = |s(f_k)|^2 \Delta f$ is the power allocation at frequency f_k. It is important to understand that this type of design does not constrain the phase of the signal, therefore it opens the way to incorporating other constraints on the transmitted signal, such as low peak to average.

We define the array gain by

$$min_{j\neq\ell} \frac{|g_{\ell,\ell}(f_k)|^2}{|g_{\ell,j}(f_k)|^2} \tag{11}$$

There are two limiting cases. The first is when the array gain for each target is sufficiently large so that the received beam contains only the desired signal and the Gaussian noise of the clutter. The second is when the main interference is caused by other targets inside the field of view. In the latter case the gain in designing the signal is less substantial since the expression in the denominator is dominated by a term which is linear in the waveform and therefore the waveform is cancelled as long as the signal to noise ratio of all targets is positive. Therefore we shall assume that the array gain is sufficient for suppressing interfering targets. In this case we would like to maximize for each ℓ

$$I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{p}\right) = \Delta f \sum_{k=1}^{K} \log\left(1 + \frac{\sigma^2_{h_\ell}(f_k) p_k |g_{\ell,\ell}(f_k)|^2}{\sigma^2_{\nu'_\ell}(f_k)}\right) \tag{12}$$

Note however that for each target we have a different cost function, and a waveform that is good for one beam is not necessarily good for another. This situation is equivalent to the concept of rate region in multiuser communication where a single node transmits simultaneously to independent nodes. To overcome this we can try to find all L-tuples of mutual informations between targets and their respective beams. To that end we define the linearly weighted sum of mutual informations by

$$I(\mathbf{p}|\boldsymbol{\alpha}) = \sum_{\ell=1}^{L} \alpha_\ell I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{p}\right) \tag{13}$$

where $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_L]^T$ is a vector of positive weights and $\|\boldsymbol{\alpha}\| = 1$. Our waveform design problem with weight vector $\boldsymbol{\alpha}$ can now be formulated as

$$\begin{array}{l} \max_{\mathbf{p}} I(\mathbf{p}|\boldsymbol{\alpha}) \\ \text{subject to } \sum_{k=1}^{K} p_k \leq P \end{array} \tag{14}$$

or more explicitly

$$\begin{array}{l} \max_{\mathbf{p}} \Delta f \sum_{\ell=1}^{L} \sum_{k=1}^{K} \alpha_\ell \log\left(1 + \frac{\sigma^2_{h_\ell}(f_k) p_k |g_{\ell,\ell}(f_k)|^2}{\sigma^2_{\nu'_\ell}(f_k)}\right) \\ \text{subject to } \mathbf{1}^T\mathbf{p} \leq P \end{array} \tag{15}$$

where $\mathbf{1} = [1, ..., 1]^T$ is a K-dimensional vector of all ones. In the next section we will describe an algorithm to perform the optimizaion problem (15).

The choice of $\boldsymbol{\alpha}$ is an interesting problem related to dynamic management of radar resources and targets priroritization. We will not pursue this issue here.

C. Multiple waveforms for spatially unresolved targets

We now extend our work to the case of multiple unresolved targets, and the design of multiple transmitted signals. We begin with revising the received signal model. Assume that an array with p elements transmits simultaneously L many waveforms. The transmitted signal at frequency f_k is given by

$$\mathbf{t}(f_k) = \sum_{l=1}^{L} \mathbf{u}_\ell(f_k) s_\ell(f_k), \qquad , k = 1, ..., K \tag{16}$$

where $\mathbf{u}_\ell(f_k)$ are the beamformer coefficients for the ℓ'th waveform designed for the ℓ'th target at frequency f_k, and $s_\ell(k)$ is the corresponding waveform at frequency f_k. We assume channel reciprocity, i.e., that if the receive steering vector is $\mathbf{a}(\theta_\ell, f_k)$ then the transmitted signal arrives at the target with channels $\mathbf{a}^*(\theta_\ell, f_k)$. The signal reflected from the ℓ'th target having signature $\mathbf{h}_\ell = \langle h_\ell(f_k), k = 1, ..., K\rangle$ is therefore given by

$$\mathbf{y}_\ell(f_k) = \sum_{m=1}^{L} \left(\mathbf{a}^*(\theta_\ell, f_k)\mathbf{u}_m(f_k)\right) h_\ell(f_k) s_\ell(f_k) \tag{17}$$

for $k = 1, ..., K$. Hence the received signal at the array is given by

$$\mathbf{x}(f_k) = \sum_{m=1}^{L} \mathbf{R}(f_k)\mathbf{u}_m(f_k) s_m(f_k) + \boldsymbol{\nu}(f_k) \tag{18}$$

where

$$\mathbf{R}(f_k) = \sum_{\ell=1}^{L} \mathbf{R}_\ell(f_k) \tag{19}$$

$\mathbf{R}_\ell$ is the rank one matrix given by

$$\mathbf{R}_\ell(f_k) = h_\ell(f_k)\mathbf{a}(\theta_\ell, f_k)\mathbf{a}^*(\theta_\ell, f_k) \tag{20}$$

As before assume that a beamformer $\mathbf{w}_\ell(f_k)$ is used to receive the ℓ'th target resulting in

$$\begin{array}{rcl} z_\ell(f_k) & = & \mathbf{w}^*_\ell(f_k)\mathbf{x}(f_k) \\ & = & \mathbf{w}^*_\ell(f_k) \sum_{m=1}^{L} \mathbf{R}(f_k)\mathbf{u}_m(f_k) s_m(f_k) + \nu'_\ell(f_k) \end{array} \tag{21}$$

where $\nu'_\ell(f_k) = \mathbf{w}^*_\ell(f_k)\boldsymbol{\nu}(f_k)$ is the received noise and clutter component of the ℓ'th beam. Let $\sigma^2_{\nu'_\ell}(f_k) = E\left|\nu'_\ell(f_k)\right|^2 \Delta f$ be the ℓ'th beam noise power at frequency f_k. To obtain the mutual information between the ℓ'th received beam and the ℓ'th target we rewrite (21) as

$$\begin{array}{rcl} z_\ell(f_k) & = & \mathbf{w}_\ell^*(f_k)\mathbf{R}_\ell \mathbf{u}_\ell(f_k) \\ & + & \sum_{m\neq\ell} \mathbf{w}_\ell^*(f_k)\mathbf{R}_m(f_k)\mathbf{u}_m(f_k) + \nu'_\ell(f_k). \end{array} \tag{22}$$

In this paper we shall not discuss the adaptive design of the beamforming vectors $\mathbf{w}_\ell^*(f_k), \mathbf{u}_m(f_k)$ but we will assume that they are given and optimize the transmitted waveforms. However we assume that the radar allocates a beam towards each target and does not perform non-linear processing jointly on the received beams $z_\ell(f_k)$ for different ℓ's. Hece the mutual information of $\mathbf{z}_\ell = \langle z_\ell(f_k) : k = 1, ..., K\rangle$ and the ℓ'th target signature $\mathbf{h}_\ell = \langle h_\ell(f_k) : k = 1, ..., K\rangle$ is given by

$$\begin{array}{l} I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{P}\right) = \\ \Delta f \sum_{k=1}^K \log\left(1 + \frac{p_{\ell,k}|g_{\ell,\ell}|^2}{\sum_{m\neq\ell} g_{n,m}(f_k)p_{m,k} + \sigma^2_{\nu'_\ell}(f_k)}\right) \end{array} \tag{23}$$

where

$$\begin{array}{l} g_{n,m}(f_k) = \mathbf{w}_n^*(f_k)\mathbf{R}_m(f_k)\mathbf{u}_m(f_k) \\ p_{n,k} = |s_n(f_k)|^2 \Delta f, \end{array}$$

$$\mathbf{p}_n = \left[p_{n,1}, ..., p_{n,K}\right]^T \tag{24}$$

is the power allocation for the n'th target and

$$\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_L]$$

is the total power allocation matrix. Finally we should notice that the PSD of the target signature is implicitly included in $g_{m,n}$.

Similarly to the single waveform case we would like to optimize a linear combination of the mutual informations. The optimization prolem can now be posed as

$$\begin{array}{l} \max_{\mathbf{P}} \sum_{\ell=1}^L I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{P}\right) \\ \text{subject to } \sum_{\ell=1}^L \sum_{k=1}^K p_{\ell,k} \le P_{\max} \end{array} \tag{25}$$

III. Waveform optimization for single and multiple targets

We now turn to the solution of (15). To that end we note that since each of the terms in the sum is a concave function of the signal power at the relevant frequency. Therefore since all coefficients are positive the cost function $I(\mathbf{p}|\boldsymbol{\alpha})$ is also concave. Furthermore the constraint is linear and hence this can be posed as a convex optimiazation by translating the problem to

$$\begin{array}{l} \min_{\mathbf{p}} -\Delta f \sum_{\ell=1}^L \alpha_\ell I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{p}\right) \\ \text{subject to } \mathbf{1}^T\mathbf{p} \le P \end{array} \tag{26}$$

The convex nature of the problem enable us to use Lagrange duality [9]. Writing the Lagrangian of the problem we obtain

$$L(\mathbf{p}, \lambda) = -\Delta f \sum_{\ell=1}^L \alpha_\ell I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{p}\right) + \lambda\left(\mathbf{1}^T\mathbf{p} - P\right) \tag{27}$$

The Lagrangian dual function is now given by

$$L_d(\lambda) = \inf_{\mathbf{p}}\left(-\Delta f \sum_{\ell=1}^L \alpha_\ell I\left(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{p}\right) + \lambda\left(\mathbf{1}^T\mathbf{p} - P\right)\right) \tag{28}$$

Since the problem is convex we have a zero duality gap which means that the solution to the dual problem

$$\begin{array}{l} \max L_d(\lambda) \\ \text{subject to} \lambda \ge 0 \end{array} \tag{29}$$

or more explicitly

$$\begin{array}{l} \max_\lambda \min_{\mathbf{p}} L(\mathbf{p}, \lambda) \\ \text{subject to} \lambda \ge 0 \end{array} \tag{30}$$

achieves the same optimal value as the primal problem. Furthermore following the KKT conditions the solution to the primal problem is given by the vector $\mathbf{p}^*$ which minimizes the Lagrangian for the optimal λ^* solving the dual problem.

Furthermore note that the Lagrangian can be writen as

$$L(\mathbf{p}, \lambda) = -\sum_{k=1}^K \Delta f \sum_{\ell=1}^L \alpha_\ell L_k\left(p_k, \lambda\right) - \lambda P \tag{31}$$

where

$$L_k\left(p_k, \lambda\right) = -\Delta f \sum_{\ell=1}^L \alpha_\ell I\left(h_\ell(f_k); z_\ell(f_k) | p_k\right) + \lambda p_k \tag{32}$$

Therefore given λ, the optimal value of $\mathbf{p}$ minimizing the dual Lagrangian function is computed coordinatewise across frequencies, transforming (30) into

$$p_k = \arg\min_p L_k\left(p, \lambda\right) \tag{33}$$

Hence we have divided the high dimensional problem into an unconstrained search over the Lagrange multiplier and multiple one-dimensional unconstrained optimization problems for each frequency in order to evaluate the dual Lagrange function. Furthermore since λ is determined by the total power constraint it can be evaluated very efficiently using a bisection method that has an exponential convergence. This is done by noting that increasing λ reduces all p_k since large values of p_k increase the Lagrangian. We begin with $\lambda = 0$ and if the total power constraint is not met we increase λ until we find a feasible solution. This is computationally very attractive.

We now discuss the multiple waveform design problem. Unlike the problem (15) we cannot reduce (25) to a convex optimization problem. However due to the special structure as a sum of functions each depending on different variables, strong duality still holds (this follows from [15])). Therefore duality theory can still be applied resulting in a simple optimization, where the relative powers at each frequency are optimized independently and only one dimensional search for the single Lagrange multiplier is performed. The details of this will be given in [16]

IV. Conclusions and extensions

In this paper we have shown that radar waveform design for multiple target estimation can be done using a linear combination of mutual informatons between each target signal and the related received beam. We have then devised a computationally efficient algorithm for solving the problem in the case of a single waveform. In future work [16] will will show how to apply duality theory to the problem of design of multiple transmit waveforms for multiple receive beams under joint power constraint. We will also show that minimizing the MSE criterion can be efficiently solved using similar optimization techniques and the design and tracking of the beamformers $\mathbf{w}_\ell(f_k)$ and $\mathbf{u}_\ell(f_k)$.

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