Optimal approach to quantum communication using dynamic programming

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Reliable preparation of entanglement between distant systems is an outstanding problem in quantum information science and quantum communication. In practice, this has to be accomplished by noisy channels (such as optical fibers) that generally result in exponential attenuation of quantum signals at large distances. A special class of quantum error correction protocols, quantum repeater protocols, can be used to overcome such losses. In this work, we introduce a method for systematically optimizing existing protocols and developing more efficient protocols. Our approach makes use of a dynamic programming-based searching algorithm, the complexity of which scales only polynomially with the communication distance, letting us efficiently determine near-optimal solutions. We find significant improvements in both the speed and the final-state fidelity for preparing long-distance entangled states.

entanglement | optimal control | quantum information | quantum repeater

Sequential decision-making in probabilistic systems is a widely studied subject in the field of economics, management science, and engineering. Applications range from problems in scheduling and asset management to control and estimation of dynamical systems (1). In this article we use these techniques for solving a class of decision-making problems that arise in quantum information science (2, 3). Specifically we consider the optimal design of a so-called quantum repeater for quantum communication. Such repeaters have potential application in quantum communication protocols for cryptography (4–6) and information processing (7), where entangled quantum systems located at distant locations are used as a fundamental resource. In principle, this entanglement can be generated by sending a pair of entangled photons through optical fibers. However, in the presence of attenuation, the probability of success in preparing a distant entangled pair decreases exponentially with distance (8).

Quantum repeaters can reduce such exponential scaling to polynomial scaling with distance and thus provide an avenue to long-distance quantum communication even with fiber attenuation. The underlying idea of quantum repeater (9, 10) is to *generate* a backbone of entangled pairs over much shorter distances, store them in a set of distributed nodes, and perform a sequence of quantum operations with only a finite probability of success. *Purification* operations (11, 12) improve the fidelity of the entanglement in the backbone, and *connection* operations join two shorter-distance entangled pairs of the backbone to form a single, longer-distance entangled pair. By relying on a quantum memory at each node to let different sections of the repeater reattempt failed operations independently, a high-fidelity entangled state between two remote quantum systems can be produced in polynomial time. A quantum repeater *protocol* is a set of rules that determine the choice and ordering of operations based on previous results. An optimal protocol is one that produces entangled pairs of a desired fidelity in minimum time within the physical constraints of a chosen implementation.

The complexity of finding the optimal repeater protocols can be understood by the following analogous example problem (1): given a sequence of rectangular matrices $M_1M_2 \ldots M_n$, such that M_k is $d_k \times d_{k+1}$ dimensional, find the optimal order of multiplying the matrices such that the number of scalar multiplications is minimized. This is a typical example of a nesting problem, in which the order in which operations are carried out affects the efficiency. For example, if $M_1 = 1 \times 10$, $M_2 = 10 \times 1$, and $M_3 = 1 \times 10$, then $(M_1M_2)M_3$ takes only 20 scalar operations, and $M_1(M_2M_3)$ requires 200 scalar multiplications. A brute-force enumeration of all possible nesting strategies and evaluation of their performance is exponential in n. To solve this problem more efficiently, we observe that the optimal nesting strategy $(M_1 \ldots (\ldots) \ldots M_p)(M_{p+1} \ldots M_n)$ should carry out the solution to its subparts optimally, that is, the nesting $(M_1 \ldots (\ldots) \ldots M_p)$ should represent the best nesting strategy for multiplying $M_1M_2 \ldots M_p$. This is the well known dynamic programming strategy (1), in which one seeks to optimize a problem by comparing different, already optimized subparts of the problem. Dynamic programming enables us to find the optimal solution to the original problem in time that is polynomial in n.

Quantum repeaters also have a nested (self-similar) structure, in which shorter-distance entanglement is used to create longer-distance entanglement, which is then used in turn for further extending the distance between entangled pairs. This structure allows us to use the methods of dynamic programming to find optimal nesting strategies for designing quantum repeater protocols.

We now proceed to detail the specific optimization problem, then discuss our dynamic programming solution to the problem. We next examine two representative schemes that we wish to optimize [the scheme of Briegel and colleagues (BDCZ scheme) in refs. 9 and 10, and the scheme of Childress *et al.* (CTSL scheme) in refs. 13 and 14], and find significant improvements in both preparation time and final fidelity of long-distance entangled pairs.

Dynamic Programming Approach

General Quantum Repeater Protocol. Quantum repeater protocols have a *self-similar structure*, where the underlying operations at each stage of the repeater have the same basic algorithms. In other words, the structure of the problem remains the same at each stage, but the parameters can be different. A generic quantum repeater consists of three kinds of operations: entanglement generation, entanglement connection, and entanglement purification. Entan-

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Abbreviations: BDCZ, Briegel–Dur–Cirac–Zoller; CTSL, Childress–Taylor–Sorensen–Lukin; DLCZ, Duan–Lukin–Cirac–Zoller.

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gled pairs are first generated and stored over a short distance, L_0. At the first nesting level, two entangled pairs of distance L_0 can be extended to distance $L_1 \approx 2L_0$ by means of an entanglement connection (6). Because of the limited fidelity of the short pairs and the imperfections from the connection operations, the fidelity of the longer pair produced by connection is generally lower than that of the shorter one. Nevertheless, the fidelity of the longer pair can be improved by entanglement purification, which is able to extract high-fidelity entangled pairs from an ensemble of low-fidelity ones by using operations that are local (restricted to qubits within a given node) (11, 12). An efficient approach of entanglement purification is entanglement pumping (9, 10), which purifies one and the same pair by using low-fidelity pairs with constant fidelity.‖ Thus, at the $(k + 1)$th nesting level, the three underlying operations (preparation at distance L_k, connection, and purification) lead to preparation at a distance $L_{k+1} \approx 2L_k$.**

Inductive Optimization. We now define the optimization problem. ***Definition.*** For given physical resources, desired distance, L_{final}, and final fidelity, F_{final}, an *optimal protocol* minimizes the expected time to have an entangled pair of fidelity, $F \geq F_{\text{final}}$, at a distance $L \geq L_{\text{final}}$.

To solve this optimization problem, the choice of parameters for the quantum operation cannot be viewed in isolation; one has to trade off the desirability of low present cost (in terms of time) with the undesirability of high future costs. If one tries to enumerate and test all possible adjustable parameters, the complexity to search for the optimized implementation scales at least exponentially with the number of repeater nodes. A simple example is provided if we make our only adjustable parameter the choice between zero and one purification step at each stage of the protocol. For the BDCZ scheme with $128 + 1$ repeater nodes, there are already $2^{128} \gtrsim 10^{38}$ possibilities, which is beyond the capability of current computers. Thus, a systematic searching method is needed to find the optimized implementation of such a huge parameter space.

Based on the self-similar structure above, we may express the optimized protocol to produce long entangled pairs in terms of a set of optimized protocols for producing shorter pairs. The general searching procedure can be performed inductively, as detailed in the following list.

1. Find and store implementations that optimize the average time (for all fidelities $f_1, \ldots, f_q$) with distance $= n = 1$, taking $\mathcal{O}(q)$.
2. Assume known optimized implementations (for all fidelities) with distance $\leq n$.
3. Find optimized implementations to produce unpurified pairs (for all fidelities) with distance $= n + 1$ by trying (connecting) all combinations of known optimized implementations with distance $\leq n$, with complexity of order $\mathcal{O}(q^2 n)$.
4. Find optimized implementations to produce purified pairs (for all fidelities) with distance $= n + 1$ by trying all combinations of unpurified pairs with distance $= n + 1$, pumping for $m = 0, 1, 2, 3, \ldots$ times; complexity goes as $\mathcal{O}(m_{\max} q^2)$.
5. Store the optimized implementations (for all fidelities) with distance $= n + 1$, based on step 4.
6. Replace n by $n + 1$, and go to step 2.

We make a discrete set of target fidelities [see supporting information (SI) *Methods* for details], $F = \{f_1, f_2, \ldots, f_q\}$, such that only a finite number of different optimal protocols with shorter distances need to be developed. The complexity for each step of our optimization procedure is shown in the list; the full procedure scales as $\mathcal{O}(q^2 n^2)$, where the $\mathcal{O}(n)$ repetitions of step 3 take the most time.‡‡ In practice, we found the full search of step 3 to be unnecessary; the search can be restricted to pairs of distance $n/2 \pm \mathcal{O}(\log(n))$, leading to complexity $\mathcal{O}(q^2 n \log(n))$.

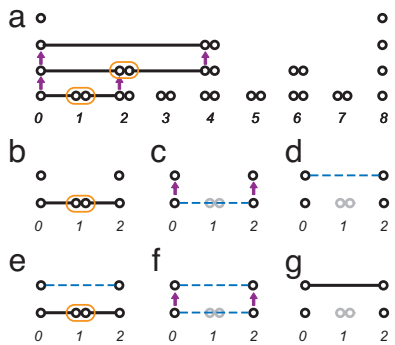

Fig. 1. Quantum repeater scheme from refs. 9 and 10 (BDCZ scheme). (*a*) In a typical realization with $n + 1 = 9$ nodes, the number of qubits per node is bounded by $2 \log_2 2n = 8$. (*b–d*) Two entangled pairs with distance 1 are connected (orange oval) at node 1 to produce an entangled state with distance 2, which is stored (purple arrows) in the qubits at higher level. (*e–g*) Another entangled state with distance 2 is produced to purify (purple arrows) the entangled state stored in qubits at higher level. Similarly, entangled states with distance 2^n can be connected to produce entangled state with distance 2^{n+1}, which may be further purified, as indicated in *a*.

Repeater Schemes and Physical Parameters. So far, we have only taken a general perspective in explaining quantum repeater protocols and describing the procedure of inductive searching by using dynamic programming. In this subsection, we specify the parameters to be optimized by examining the schemes of the quantum repeater, physical restrictions on entanglement generation for current techniques, and the error models of local quantum gates. Only with a functional relationship between physically adjustable parameters and repeater operation outputs can we find the optimized implementations for procedures 1, 3, and 4 in the list above.

There are several different *schemes* for building a quantum repeater that differ primarily in the amount of physical resources utilized. For example, in the BDCZ scheme (9, 10) (Fig. 1), the maximum number of qubits in the quantum memory (to store intermediate states for connection and purification) required for each repeater node increases logarithmically with the total number of repeater nodes. In the CTSL scheme (13, 14) (Fig. 2), an efficient way to use quantum memory is proposed, and only *two* qubits are needed for each node, regardless of the total number of repeater nodes. One of the two qubits is called the *communication qubit*, which is optically active and can be used to generate entanglement with other communication qubits from neighboring nodes. The other qubit is called the *storage qubit*, which can be used to store quantum state over very long time. As shown in Fig. 2*b*, with the help of local gates (orange solid ovals) between communication and storage qubits, the entangled state between communication qubits can be used to implement teleportation-based gates (e.g., the Controlled NOT gate) between storage qubits from neighboring nodes (7, 15). Such remote gates (orange dashed ovals) are sufficient for entanglement connection and purification of the storage qubits; communication qubits are providing the necessary resource mediating the gates between remote storage qubits. For clarity, we will omit the communication qubits in the following discussion but

‖In principle, repeater schemes exist (see ref. 10 and references therein) that work much faster. For those schemes, however, the number of memory qubits per repeater station scales at least linearly with the final distance, which make them impractical.

**Because we must wait for entanglement generation and purification to succeed before proceeding to the next nesting level, the overall time for successful pair generation, in general, is much longer than that of classical communication over the given distance.

‡‡For the CTSL scheme, there will be another $\mathcal{O}(q)$ overhead, associated with $\mathcal{O}(q)$ possible fidelity choices of the entangled state for the Controlled NOT gate (see Fig. 2*b*).

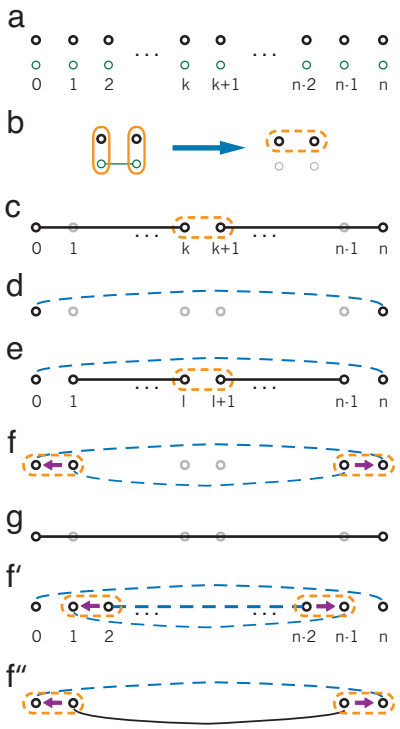

Fig. 2. Quantum repeater scheme from refs. 13 and 14 (CTSL scheme). (*a*) This scheme has exactly two qubits per node. The communication qubits (green nodes) are used for entanglement generation and short-term storage; the storage qubits (black nodes) are used for long-term storage. (*b*) With the help of local gates (orange solid ovals) between communication and storage qubits, the entangled state between communication qubits can be used to implement entangling gates (e.g., the Controlled NOT gate) between storage qubits from neighboring nodes. The effective remote gate is highlighted by the orange dashed oval. Such remote gate is sufficient for entanglement connection and purification of storage qubits. The communication qubits are omitted in the following plots. (*c* and *d*) Entanglement connection to produce an unpurified entangled state with distance *n*. (*e*–*g*) Entanglement connection to produce an unpurified entangled state with distance $n-2$ to purify the entangled state with distance *n*. (*f*′ and *f*″) Illustration of multilevel pumping. An entangled state with distance $n-4$ is used to purify an entangled state with distance $n-2$, and the (purified) latter is used to purify an entangled state with distance *n*. The only difference between *f* and *f*′ is the fidelity of the entangled state with distance $n-2$. The latter has higher fidelity.

still keep track of the mediated operation between remote storage qubits.

To model errors in the physical operations, we need to introduce a number of parameters determined by the quantum hardware. For entanglement generation, the relationship between the fidelity of the elementary pair, F_0, and the generation time, τ_e, depends on the physical parameters (such as the signal propagation speed, c, the fiber attenuation length, L_{att}, the efficiency of single photon collection and detection, ε, and the distance of elementary pair, L_0) and the specific approach to generate entanglement.[§§]

For entanglement connection and pumping, the dominant imperfections are errors from measurement and the local two-qubit gate, which we model with a depolarizing channel. In particular, the model for measurement is quantified by a reliability parameter (9, 10), η, which is the probability of faithful measurement. For example, a projective measurement of state $|0\rangle$ would be

$$P_0 = \eta|0\rangle\langle 0| + (1-\eta)|1\rangle\langle 1|.$$

Similarly, the model for the local two-qubit gate is characterized by a reliability parameter (9, 10), p. With probability, p, the correct operation is performed; otherwise the state of these two qubits is replaced by the identity matrix (9, 10). For example, the action on a two-qubit operation, U_{ij}, would be

$$U_{ij}\rho U_{ij}^{\dagger} \rightarrow pU_{ij}\rho U_{ij}^{\dagger} + \frac{1-p}{4} Tr_{ij}[\rho] \otimes I_{ij},$$

where $\mathrm{Tr}_{ij}[\rho]$ is the partial trace over the subsystem i and j, and I_{ij} is the identity operator for subsystem i and j. In general, the reliability parameters (η and p) should be reasonably high [i.e., above some thresholds (9, 10)], so that the suppression of error from entanglement pumping dominates the new errors introduced by entanglement connection and entanglement pumping.[¶¶]

Optimization Parameters. We now list the adjustable parameters we can optimize during procedures 1, 3, and 4 in the list above.

1. During the entanglement generation, there is freedom to choose the generation time τ_e, which is determined by the success probability and the communication time. In general, the higher the success probability, the shorter the generation time and the lower the fidelity of the entangled state, so the generation time and the fidelity should be balanced (13, 14).
2. During the entanglement connection, the distances of two shorter pairs can be adjusted, and the total distance is kept unchanged.
3. During entanglement purification, the number of steps is also adjustable, which should balance the gain in fidelity and the overhead in time.

Additional Operations. Besides the operations from the original quantum repeater schemes, some *additional* operations might be useful. For example, we may skip several intermediate repeater nodes (*node skipping*) to generate entanglement between distant nodes directly with a substantially lower success probability. Also, during entanglement pumping, we might consider *multilevel pumping* (16), in which several levels of entanglement pumping are nested together before the next level of entanglement connection (Fig. 2*f*″). Multilevel pumping can produce entangled pairs with higher fidelity than single-level pumping. Such additional operations can be easily incorporated into search procedures 1, 3, and 4. We will show that the dynamic programming approach can use these additional operations appropriately to reduce the average time, extend the upper bounds for achievable final fidelity, and even improve the threshold for the reliability parameters of p and η.

Results and Discussion

Improvement of BDCZ and CTSL Schemes. With procedures as listed above, we implemented a computer program to examine the mean time to prepare entangled pairs and to search according to our dynamic programming prescription through the parameter space outlined earlier. We looked for optimal protocols for a quantum repeater for all distances $\leq$1,280 km and target fidelities $\geq$0.8. Unless otherwise specified, we use $L_{att} = 20$ km, $\varepsilon = 0.2$, and $\eta = P = 0.995$ for the rest of the discussion. We first fix $L_0 = 10$ km, and we will consider the optimization of L_0 to justify such a choice later. To visualize the results, the profile of the optimized time (smooth

[§§]For example, the entanglement generation approach using scattering as proposed in refs. 13 and 14 would be

$$F_0 = F_0(\tau_e) = \frac{1}{2}\left\{1 + \left[1 - \frac{L_0}{\tau_e c} e^{L_0/L_{att}}\right]^{2(1-\varepsilon)/\varepsilon}\right\}$$

[¶¶]We neglect the time associated with local operations, which is usually much shorter than the communication time between neighboring repeater stations. Nonnegligible gate operation time can easily be included in our optimization.

PHYSICS

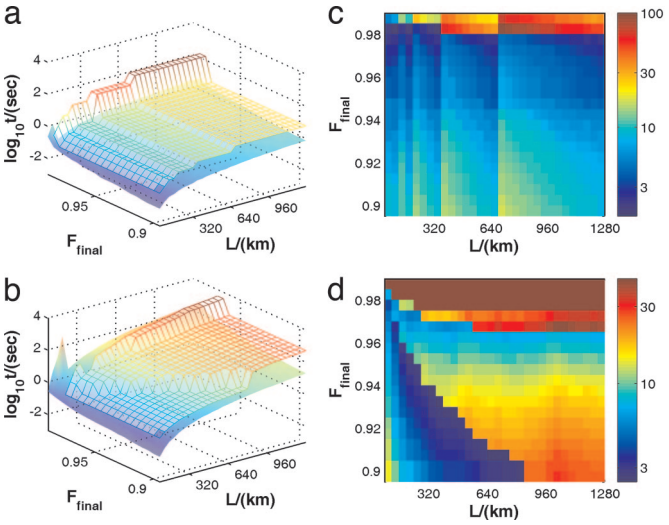

Fig. 3. Plots of time profiles and improvement factors. (*a* and *b*) Speedup in time associated with various final distances and fidelities. Shown are t (F_{final}, L) for unoptimized (meshes) and optimized (smooth surface) implementations of the BDCZ scheme (*a*) and of the CTSL scheme (*b*). (*c* and *d*) Pseudocolor plots of the improvement factor, $t_{\text{unopt}}/t_{\text{opt}}$, for the BDCZ scheme (*c*) and for the CTSL scheme (*d*), in the region ($F_{\text{final}} > 97.5$), the improvement factor $t_{\text{unopt}}/t_{\text{opt}} \to \infty$. The default parameters are $L_{\text{att}} = 20$ km, $\varepsilon = 0.2$, and $p = \eta = 0.995$.

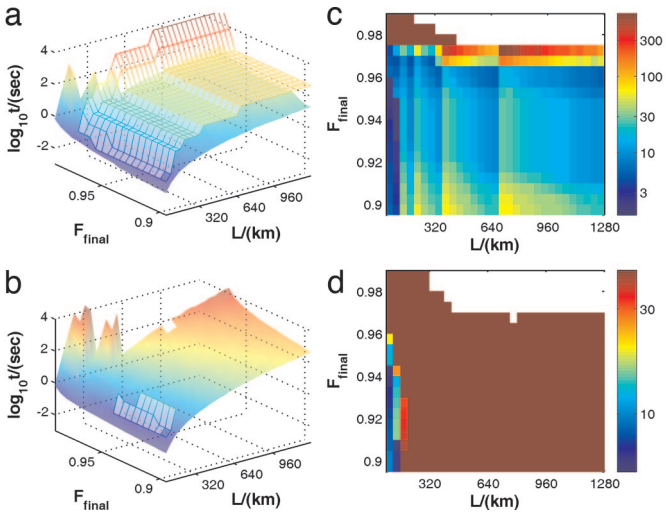

Fig. 4. Plots of time profiles and improvement factors. The subplots are arranged in the same way as Fig. 3. Local operations have lower reliability parameters, $p = \eta = 0.990$. (*a* and *c*) For the BDCZ scheme, the optimization procedure only improves the speed of the quantum repeater and does not extend the achievable region in the *F–L* plot. (*b* and *d*) For the CTSL scheme, for distances >200 km, the improvement factor, $t_{\text{unopt}}/t_{\text{opt}} \to \infty$. Here, the reliability parameter ($p = \eta = 0.990$) is *insufficient* to create distant entangled pairs with the unoptimized implementation, but the optimized implementation (with multilevel pumping) is still able to create high-fidelity distant entangled pairs, because multilevel pumping lowers the threshold of the reliability parameters for the CTSL scheme.

surface) is plotted in Fig. 3 *a* and *b* with respect to the final distance (from 10 to 1,280 km) and the fidelity (from 0.90 to 0.99) for both the BDCZ and the CTSL schemes. The calculation optimizes over the elementary pair generation (both distance and generation time), the connecting positions, and the number of pumping steps, with spacing between neighboring repeater nodes of 10 km; both additional operations (node skipping and multilevel pumping) are also included for the optimization. For comparison, the unoptimized time profiles (meshes) for the BDCZ and the CTSL schemes are also plotted. The unoptimized protocol assumes fixed elementary pair fidelity ($F_0 = 0.96$ and 0.99 for BDCZ and CTSL, respectively), simple connection patterns (detailed in refs. 9, 10, 13, and 14), and a constant number of pumping steps.

As expected, the unoptimized protocol always takes a longer time than the optimized protocol for the same final distance and target fidelity. Time profiles for the unoptimized protocols have *stairlike jumps* in Fig. 3 *a* and *b*. For the BDCZ scheme (Fig. 3*a*), the jumps occurring with increasing distance (occurring at distances $L/L_0 = 2^p + 1 = 1, 3, 5, 9, 17, 33, \ldots$) are the results of time overhead from the additional level of connection; the jump occurring at $F_{\text{final}} = 0.98$ is due to the sudden change in the number of pumping steps from 1 to 2. Similarly, for the CTSL scheme (Fig. 3*b*), the two jumps are due to the change of the number of pumping steps from 1 to 2 and finally to 3. For the optimized protocols, the time increases smoothly with increasing final distance and fidelity.

The improvement factor (i.e., the ratio between the times for unoptimized and optimized protocols) is plotted for both the BDCZ and the CTSL schemes in Fig. 3 *c* and *d*. As we might expect, the previously mentioned jumps lead to sharp *stripes* where the improvement factor changes discontinuously. There are several regions where the optimization gives significant improvement. For example, for the BDCZ scheme, the vertical bright stripes indicate that the optimization provides a time-efficient way to generate entangled pairs for distance $(2^p + \delta_+)\, L_0$ (with $\delta_+ > 0$), gaining a factor of ≈10; the horizontal bright stripes indicate that efficiently arranging the number of pumping steps can also speed up the scheme by a factor of ≈30 or even more. For most of the optimized protocols, a distant pair is divided into two shorter pairs with similar distance and fidelity (symmetric partition), but occasional asymmetric partitioning can further reduce the time by ≈10%.

For the BDCZ scheme, the correspondence between jumps and stripes essentially accounts for all of the features of the improvement plot (Fig. 3*c*). For the CTSL scheme, however, besides the stripes, there is also a region (with distance $L > 100$ km and fidelity $F > 97.5\%$) where the improvement factor is infinity: optimization not only boosts the speed, but also extends the upper bound of achievable fidelity for distant pairs.

We also study the improvement for other choices of reliability parameters, p and η, especially those values close to the threshold (9, 10). Suppose the reliability parameters are $p = \eta = 0.990$. In Fig. 4 *a* and *c*, we plot the speedup in time associated with various final distances and fidelities for the BDCZ scheme. For both (optimized and unoptimized) protocols, the highest achievable fidelity is ≈97.5% (compared with 99% in Fig. 3*c*), limited by errors from local operations. The improvement factor ranges between [1.5, 600] (compared with [1, 100] in Fig. 3*c*). Apart from these differences, the key features (horizontal and vertical stripes) of improvement from optimization are very similar between Figs. 3*c* and Fig. 4*c*.

For the CTSL scheme, however, unoptimized and optimized protocols behave very differently, when $p = \eta = 0.990$. As shown in Fig. 4 *b* and *d*, the unoptimized protocol cannot effectively create entangled pairs for distances >200 km, whereas the optimized protocol is still able to efficiently create distant entangled pairs with very high fidelity. Thus our optimization lowers the threshold requirement for the CTSL scheme of quantum repeater.

To understand the reason for the improvement of the highest achievable fidelity (Fig. 3*d*) and the parameter threshold (Fig. 4*d*), we examine the optimized protocol of the CTSL scheme in the next two subsections.

Comparison Between Optimized and Unoptimized Protocols. We first compare the detailed procedures between two (optimized and unoptimized) protocols by using the CTSL scheme to produce a pair with final distance $L = 11L_0$ and fidelity $F_{\text{final}} = 97.6\%$, with

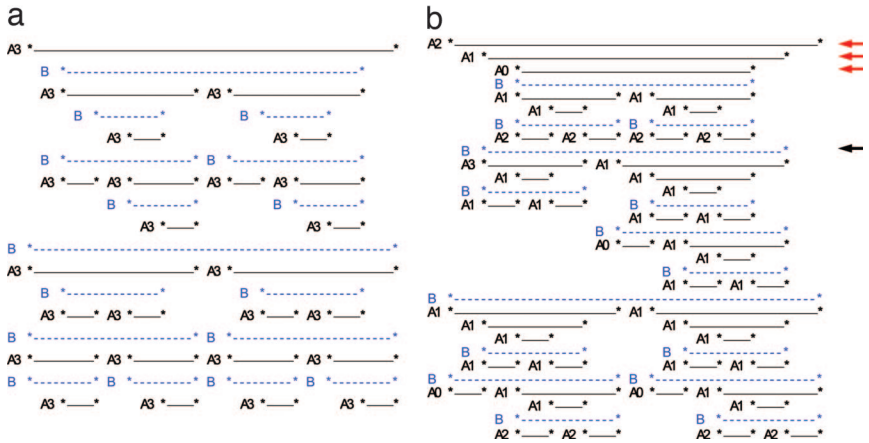

Fig. 5. Two implementations with targeting final distance $L = 11L_0$ and fidelity $F_{\text{final}} = 0.976$, by using the CTSL scheme. Each storage qubit is represented by $*$. All of the relevant entangled states are shown. The order to produce these entangled states is from bottom to top; states on the same row can be produced simultaneously. The two kinds of entangled states are: purified entangled states (type A, solid black line) and unpurified entangled states (type B, dashed blue line). On the left side of each purified entangled state, there is a label ''Ak,'' where the number k indicates that this purified entangled state is obtained from k steps of entanglement pumping. (*a*) The unoptimized (*Left*) implementation has three pumping steps after each entanglement connection, with average time of ≈11 sec to produce the pair wanted. (*b*) The optimized (*Right*) implementation is from optimization over pair-generation time, connection position, and the number of pumping steps. The optimized choice of connection position does not necessarily break the long pair into two almost identical shorter pairs; for example, the entangled state to which the black arrow points in the ninth row is obtained by connecting two very different shorter pairs in the row below. In addition, the possibility of multilevel pumping is also taken into account during the dynamic programming. As pointed out by the red arrows, the pair of storage qubits in the third row pumps the pair in the second row, and the latter pumps the pair in the first row. The average time is ≈1.5 sec for the optimized implementation, about 8 times faster than the unoptimized one.

default reliability parameters $p = \eta = 0.995$. We choose the highest fidelity achievable by the unoptimized protocol, so that we will see almost all features that give improvements. The results for the unoptimized protocol (Fig. 5*a*) follow refs. 13 and 14 exactly, whereas the optimized protocol (Fig. 5*b*) is from our systematic search using dynamic programming. They differ in the following aspects: (*i*) during entanglement generation, the optimized implementation generates elementary pairs with fidelity lower than 0.99 to reduce the generation time and uses entanglement pumping afterward to compensate for the fidelity loss; (*ii*) during entanglement connection, the rule of producing a long pair from two almost identical shorter pairs is slightly modified (e.g., the pair pointed to by the black arrow in the ninth row is from the connection of two quite different pairs in the tenth row); (*iii*) the number of pumping steps after each connection varies from 0 to 3 for optimized implementation; (*iv*) finally, the optimized implementation uses multilevel pumping, which is discussed in detail in the next subsection. For clarity, the additional operation of node skipping is suppressed in the optimization here. The overall average time is reduced from 11 to 1.5 sec, improved by a factor of 8.

Note that our optimization results based on average-time approximation (see Fig. 6 and *SI Methods*) are confirmed by the Monte Carlo simulation of the optimized protocols, verifying the substantial speedup.

Multilevel Pumping. We now consider the additional operation of multilevel pumping in more detail. We discuss multilevel pumping only for the CTSL scheme, not for the BDCZ scheme. (In the BDCZ scheme, introduction of multilevel pumping requires additional quantum memory qubits.) In the original unoptimized protocol (13, 14), the purified entangled state with distance n [between the 0th and the nth nodes ($n > 5$)] is produced by entanglement pumping, and the entangled states used for pumping (called pumping pairs) are *unpurified* entangled states with distance $n - 2$ (Fig. 2*f*). The fidelity of these pumping pairs with distance $n - 2$ are limited by the connection operation, which imposes an upper bound for the fidelity of the purified pair with distance n. The underlying restriction is that the pumping pair is unpurified.

We may lift this restriction by allowing the use of a purified pumping pair. This is multilevel pumping. For example, the pumping pair with distance $n - 2$ may also be produced by entanglement pumping from pumping pairs with distance $n - 4$ (Fig. 2*f′*), and so on. By doing multilevel pumping, the fidelity of the pumping pair with distance $n - 2$ is increased (Fig. 2*f′*), and the same for the fidelity upper-bound for the entangled state with distance n. Although multilevel pumping can increase the fidelity, it also slows down the repeater scheme.

When the reliability of local operations is above the threshold for the unoptimized protocol (e.g., $p = \eta = 0.995$), we find that multilevel pumping is necessary only for the last two or three levels to the high-fidelity pair we want to produce. Such multilevel

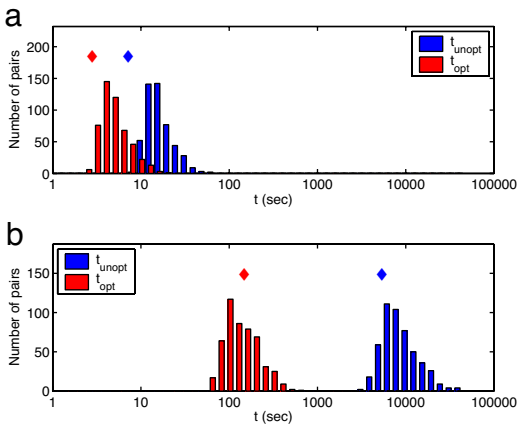

Fig. 6. Results of Monte Carlo simulation. Monte Carlo simulation for unoptimized/optimized implementations for the BDCZ scheme (*a*) and the CTSL scheme (*b*), with a final distance of 1,280 km and a fidelity of 0.97. The time distributions for distant pairs are plotted, with red (blue) bars for optimized (unoptimized) implementation. In each plot, the red (blue) diamond indicates the estimated time from average-time approximation for optimized (unoptimized) implementation. The average-time approximation provides a good estimate up to some overall factor ($2 \approx 3$), which is not very sensitive to the implementation.

pumping can be identified in the optimized implementation; for example, as indicated by red arrows in Fig. 5*b*, the pair of storage qubits in the third row pumps the pair in the second row, and the latter pumps the pair in the first row. On the other hand, when the reliability of local operations is below such a threshold (e.g., $p = \eta = 0.990$), multilevel pumping is needed after almost every entanglement connection.

If we exclude the possibility of multilevel pumping in dynamic programming, the infinite improvement factor for pairs with distance $L > 100$ km and fidelity $F \gtrsim 97.5\%$ in Fig. 3*d* would disappear. Similarly, in Fig. 4*d*, without multilevel pumping, there would be no improvement of the parameter threshold, and even the optimized protocol could not efficiently create distant ($L > 200$ km) entangled pairs. For the CTSL scheme, multilevel pumping not only enables us to prepare entangled pairs with very high fidelity, but also lowers the required threshold of the reliability parameters (p and η) for local operations. Therefore, the flexibility to include additional operations in our dynamic programming provides a new perspective on the optimization of quantum repeater schemes.

Other Improvements. In addition to the previously discussed features in the plots of improvement factor, there is an overall improvement for all final distances and fidelities. Such overall improvement comes from the optimized choice of the distance (by node skipping) and the generation time for *each* elementary pair used. Such overall improvement is ≈ 1.5 (or $2 \approx 3$) for the BDCZ (or CTSL) scheme, which indicates that the original choice of uniform distance $L_0 = 10$ km and initial fidelities $F_0 = 96\%$ (or 99%) are quite close to the optimal.

Finally, we consider whether it is possible to gain some additional speedup if we are allowed to choose the location of the nodes of the quantum repeater. To answer this question, we discretize the distance into smaller units, for example, 1 km $\ll L_{\text{att}}$. Because the distance of each elementary pair is determined by the dynamic programming, the optimized location of the nodes can be inferred from the distances of the elementary pairs. We find that the speedup due to optimization over the location of the nodes is fairly small, no more than 15% in time (for cases with final distances >200 km). In general, we find that as long as the node spacing is less than the attenuation length ($L_0 < L_{\text{att}}$), a quantum repeater can be implemented almost optimally.

Experimental Implications. Throughout our analysis we have assumed relatively high fidelity of local measurements and operations ($\eta = P = 0.995$ or 0.99) and memory times exceeding total communication times. Recent experiments with tapped ions (17, 18), neutral atoms (19), and solid-state qubits (20) are already approaching these values of fidelity and memory times. At the same time, high initial entanglement fidelity ($F_0 \approx 96\%$ or 99%) is also needed for the optimized protocols. Entanglement fidelity of $\approx$90% can be inferred from recent experiments with two ions in independent traps (21). Although optimization procedure can yield protocols compatible with fairly low initial fidelity and high local error rates, in practice, these errors introduce a large overhead in communication time.

Besides the schemes considered here, there are other quantum repeater schemes, in particular, the Duan *et al.* (22) scheme (DLCZ scheme) that requires a smaller set of quantum operations and relatively modest physical resources. The original DLCZ scheme does not use active entanglement purification and hence cannot correct arbitrary errors. In such a case, optimization is straightforward and has been discussed in ref. 22. Recently, the DLCZ scheme has been extended to include active entanglement purification to suppress, for example, phase noises (23, 24). The extended DLCZ scheme becomes very similar to the BDCZ scheme in terms of the self-similar structure. The technique of dynamic programming can be applied to optimize the extended DLCZ scheme as well.

Conclusion and Outlook

We have demonstrated how dynamic programming can be a powerful tool for systematically studying the optimization of quantum repeater protocols. We find substantial improvements to two specific repeater schemes (9, 10, 13, 14). Beyond searching for optimal choices in previously known elements of the schemes (entanglement generation, connection, and pumping), our systematic study can also incorporate more sophisticated additional operations, such as node skipping, multilevel pumping, and the flexible location of repeater stations. In particular, our multilevel pumping procedure extends the maximum achievable fidelity for distant pairs. It should be possible to include additional possibilities to the optimization problem of quantum repeater, such as different choices of entanglement generation and possibly more efficient usage of local qubits (25, 26). It would also be interesting to study the optimization problem of quantum repeater with finite storage time of the quantum memory (27, 28). Even the optimized protocols have a rather limited speed (corresponding to generation of one high-fidelity pair over 1,280 km in 1 to $\approx$100 sec (see Fig. 6). Therefore, improvement of experimental techniques (to obtain higher local-operation fidelities and more efficient atom–photon coupling) as well as development of new theoretical approaches to speed up quantum repeaters still remain an outstanding goal. Furthermore, the dynamic programming techniques may find application in other outstanding problems in quantum information science, such as the optimization of quantum error correction for fault-tolerant quantum computation. In particular, the optimization of the network-based quantum computation scheme with minimal resources (15) might be possible.

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