

# COMPRESSIVE SENSING AND WAVEFORM DESIGN FOR THE IDENTIFICATION OF LINEAR TIME-VARYING SYSTEMS

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## ABSTRACT

In this paper, we investigate the application of compressive sensing and waveform design for estimating linear time-varying system characteristics. Based on the fact that the spreading function system representation is sparse in realistic system scenarios, we propose a new method for the identification of narrowband, wideband and dispersive systems using a small set of measurements. Through numerical simulations, we successfully demonstrate the feasibility of using compressive sensing to estimate the system spreading function.

**Index Terms**— Linear time-varying system, compressive sensing, system identification

## 1. INTRODUCTION

Processing of linear time-varying (LTV) systems is important in numerous applications including radar, sonar and communications. As a result, different mathematical representations have been used to characterize LTV systems. In particular, an LTV system can be characterized by a kernel representation associated with a characteristic transform and a spreading function. The characteristic transform describes how the system affects the propagating signal, and the spreading function describes how the signal energy diffuses during propagation.

LTV systems can be classified as narrowband, wideband and dispersive. Narrowband LTV systems can be represented by the narrowband spreading function (SF) [1]; this representation has received wide application with multipath fast-fading wireless communication channels as well as radar and sonar systems. LTV systems with wideband properties have also been represented using the kernel formulation and a wideband version of the SF (WSF). Wideband LTV systems are characterized by time delay and Doppler scale changes to describe the physical effect of the system on the analysis signal [2, 3]. There are many systems in nature with dispersive time-frequency characteristics as they can cause different frequency components to be shifted by different amounts. A dispersive system output can be modeled as a superposition of instantaneous frequency shifts, weighted by a matched dispersive spreading function (DSF) [4]. Using a discretization

procedure, and under certain physical assumptions on the systems, we can obtain discrete equivalent representations for LTV systems, which are useful in real applications.

LTV system identification is equivalent to estimating the spreading function of the corresponding system. However, this is a difficult problem in general. In this paper, we propose to solve this problem by considering the fact that in many cases, due to physical restrictions on the real systems, the spreading functions of the aforementioned LTV systems are sparse. For example, mobile radio channels can be approximated as underspread as they do not introduce substantial TF shifts [5]. Recently, there has been a growing interest in recovering sparse signals from their projection onto a random vector using compressive sensing [6]. As a result, we use compressive sensing to identify LTV systems from a small set of measurements. This is potentially useful in applications where one cannot collect a lot of measurements or cannot transmit many signals.

After a short review on LTV system characterizations in Section 2, we present the SF estimation approach in Section 3. In Section 4, we present the waveform design needed for compressive sensing, and we present some simulations in Section 5.

## 2. LTV SYSTEM REPRESENTATIONS

### 2.1. LTV Systems

**Narrowband System Representations** The output of a narrowband LTV system can be represented as a superposition of time-frequency (TF) shifted versions of the input signal  $x(t)$ , weighted by the SF, i.e.,

$$(\mathcal{L}x)(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{SF}_{\mathcal{L}}(\tau, \nu) e^{-j\pi\tau\nu} (\mathcal{M}_{\nu}\mathcal{S}_{\tau}x)(t) d\nu d\tau, \quad (1)$$

where  $(\mathcal{S}_{\tau}x)(t) = x(t-\tau)$  is the time shift operation, and the frequency shift operation is given by  $(\mathcal{M}_{\nu}x)(t) = x(t) e^{j2\pi\nu t}$ .

Assuming that the input signal  $x(t)$  is bandlimited to  $[f_0, f_1]$  with bandwidth  $W = f_1 - f_0$  and that the output signal  $(\mathcal{L}x)(t)$  is time-limited to  $[t_0, t_1]$  with duration  $T = t_1 - t_0$ , (1) can be decomposed into

$$(\mathcal{L}x)(t) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \widehat{\text{SF}}_{\mathcal{L}}\left(\frac{m}{W}, \frac{n}{T}\right) x_{m,n}(t) \quad (2)$$

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where  $x_{m,n}(t) = (\mathcal{M}_{\frac{n}{T}} \mathcal{S}_{\frac{m}{W}} x)(t)$  and  $\widehat{\text{SF}}_{\mathcal{L}}(\frac{m}{W}, \frac{n}{T})$  are two-dimensional (2-D) samples of a smoothed SF [1].

**Wideband System Representations** For a wideband LTV system  $\mathcal{B}$  defined on  $L_2(\mathbb{R})$  with input  $x(t)$ , the output  $(\mathcal{B}x)(t)$  can be characterized by a superposition of time shifts and scale changes, weighted by the WSF,

$$(\mathcal{B}x)(t) = \int_0^\infty \int_{-\infty}^\infty \text{WSF}_{\mathcal{B}}(\tau, a) (\mathcal{S}_\tau \mathcal{C}_a x)(t) d\tau da. \quad (3)$$

The scale operation  $(\mathcal{C}_a x)(t) = \sqrt{|a|}x(at)$  is due to reflections off fast moving point scatterers. The discrete wideband model of (3) was obtained in [2] assuming that the Fourier transform  $X(f)$  of the input signal  $x(t)$  is bounded within  $f \in [-W/2, W/2]$ , and that its Mellin transform  $\text{MT}_x(\beta)$  is bounded within  $\beta \in [-\beta_0/2, \beta_0/2]$ . Specifically,

$$y(t) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{\chi}_{\mathcal{B}}\left(\frac{n}{a_0^m W}, a_0^m\right) a_0^{\frac{m}{2}} x\left(a_0^m t - \frac{n}{W}\right) \quad (4)$$

where the basic scaling factor is  $a_0 = e^{1/\beta_0}$ , and  $\hat{\chi}_{\mathcal{B}}(\tau, a)$  is a 2-D smoothed version of the WSF.

**Dispersive System Representations** If a dispersive system  $\mathcal{Z}$  changes the phase function of the input  $x(t)$  by  $^1 \xi(t/t_r)$ , the dispersive (time-dependent) frequency shift is given by  $\nu(t) = \frac{d}{dt} \xi(t/t_r)$ . A dispersive version of the SF (DSF) was developed to match the system dynamics based on  $\xi(t/t_r)$ . Specifically, the DSF was used to interpret the system output  $(\mathcal{Z}x)(t)$  as a weighted superposition of dispersive transformations on  $x(t)$ :

$$(\mathcal{Z}x)(t) = \int_{-\infty}^\infty \int_{-\infty}^\infty \text{DSF}_{\mathcal{Z}}(\zeta, \beta) e^{-j\pi\zeta\beta} (\mathcal{D}_\beta^{(\xi)} \mathcal{G}_\zeta^{(\xi)} x)(t) d\zeta d\beta.$$

These transformations correspond to a generalized time shift operation  $\mathcal{G}_\zeta^{(\xi)} x(t) = (\mathcal{U}_\xi^{-1} \mathcal{S}_{t_r \zeta} \mathcal{U}_\xi x)(t)$  and an instantaneous frequency shift operation  $(\mathcal{D}_\beta^{(\xi)} x)(t) = (\mathcal{U}_\xi^{-1} \mathcal{M}_{\beta/t_r} \mathcal{U}_\xi x)(t) = x(t) e^{-j2\pi\xi(\frac{t}{t_r})}$ , where  $\mathcal{M}_\nu$  and  $\mathcal{S}_\tau$  are defined in (1). The unitary warping operator  $\mathcal{U}_\xi$  is defined as

$$(\mathcal{U}_\xi x)(t) = \left| t_r \nu \left( \xi^{-1} \left( \frac{t}{t_r} \right) \right) \right|^{-1/2} x \left( t_r \xi^{-1} \left( \frac{t}{t_r} \right) \right) \quad (5)$$

where  $(\mathcal{U}_\xi^{-1} \mathcal{U}_\xi x)(t) = x(t)$ . The warping relationship assumes that  $\xi(t/t_r)$  is a one-to-one function with  $\xi^{-1}(\xi(t/t_r)) = t/t_r$ . The corresponding discrete model can be found in [4]. It is worth noting that the composite system  $\mathcal{U}_\xi \mathcal{Z} \mathcal{U}_\xi^{-1}$  is a unitary equivalent narrowband system, for which the input and output are the time warped signals,  $(\mathcal{U}_\xi x)(t)$  and  $(\mathcal{U}_\xi \mathcal{Z} x)(t)$ , respectively. Specifically,  $\text{DSF}_{\mathcal{Z}}(\zeta, \beta) = \text{SF}_{\mathcal{U}_\xi \mathcal{Z}} \left( t_r \zeta, \frac{\beta}{t_r} \right)$ .

<sup>1</sup>Here,  $t_r > 0$  is a normalized time reference

## 2.2. Matrix Formulation of LTV System Outputs

We consider the matrix formulation for discrete LTV system outputs. As an example, we provide the matrix formulation for the narrowband system in (2). Considering  $D$  samples of the TF shifted waveform  $x_{m,n}(t)$ , we obtain the vector  $\mathbf{x}_{m,n} = [x_{m,n}[1], x_{m,n}[2], \dots, x_{m,n}[D]]^T$ . The  $D \times K$  matrix  $\Phi = [\mathbf{x}_{0,0}, \dots, \mathbf{x}_{m,n}, \dots, \mathbf{x}_{M-1, N-1}]$  contains  $M$  time shifts and  $N$  frequency shifts of the TF shifted waveform when  $K = MN$ . We also concatenate the SF into the  $K \times 1 = MN \times 1$  vector  $\mathbf{H} = [\widehat{\text{SF}}_{\mathcal{L}}(0, 0), \dots, \widehat{\text{SF}}_{\mathcal{L}}(m, n), \dots, \widehat{\text{SF}}_{\mathcal{L}}(M-1, N-1)]^T$ . Hence, the system output vector can be expressed as

$$\mathbf{Y} = \Phi \mathbf{H}. \quad (6)$$

Following a similar procedure, we can obtain a similar matrix formulation for wideband and dispersive systems.

## 3. SPREADING FUNCTION ESTIMATION USING COMPRESSIVE SENSING

Using the matrix representation for a discrete LTV system in (6), we can express the system output as  $\mathbf{Y} = \Phi \mathbf{H}$ . Here,  $\mathbf{H}$  represents the spreading function that is assumed to be an  $S$ -sparse vector, and  $\Phi \in \mathbb{R}^{D \times K}$  is the matrix representation of the signal basis. Our goal is to identify the corresponding LTV system by determining  $\mathbf{H}$  from a set of  $L$  available samples (which we refer to as measurements), where  $L$  is less than  $D$ , the dimension of  $\mathbf{Y}$ . Specifically, we want to determine  $\mathbf{H}$  from  $\mathbf{s} = \mathbf{A} \mathbf{Y} = \mathbf{A} \Phi \mathbf{H}$ , by finding the measurement matrix  $\mathbf{A} \in \mathbb{R}^{L \times D}$ . We denote  $\Psi = \mathbf{A} \Phi$  so that  $\mathbf{s} = \Psi \mathbf{H}$ .

Before we proceed with the estimation of  $\mathbf{H}$ , we briefly review some important definitions and principles of compressive sensing [6, 7].

### 3.1. Compressive Sensing Principles

We start our description by reviewing some relevant principles. The basis pursuit (BP) principle is used to solve the convex programming problem:  $\min \|\mathbf{H}\|_{\ell_1}$  subject to  $\mathbf{s} = \Psi \mathbf{H}$ , where  $\|\mathbf{H}\|_{\ell_1} = \sum_i |H_i|$  denotes the  $\ell_1$ -norm. This can be done using linear programming in the real case and cone programming in the complex case [7].

Let  $\Lambda \subset \{1, \dots, d\}$  and let  $\Psi_\Lambda$  be the submatrix of  $\Psi$  consisting of the columns indexed by  $\Lambda$ . The local isometry constant  $\delta_\Lambda = \delta_\Lambda(\Psi)$  is the smallest number satisfying  $(1 - \delta_\Lambda) \|\mathbf{H}\|_2^2 \leq \|\Psi_\Lambda \mathbf{H}\|_2^2 \leq (1 + \delta_\Lambda) \|\mathbf{H}\|_2^2$  for all coefficient vectors supported on  $\Lambda$  [7]. The (global) restricted isometry constant is then defined as

$$\delta_S = \delta_S(\Psi) := \sup_{|\Lambda|=S} \delta_\Lambda(\Psi), \quad S \in \mathbb{N}.$$

Candès, Romberg and Tao proved the following recovery theorem for the BP in [7]:

**Theorem 3.1** Assume that  $\Psi$  satisfies  $\delta_{3S}(\Psi) + 3\delta_{4S}(\Psi) < 2$  for some  $S \in \mathbb{N}$ . Let  $\mathbf{H}$  be an  $S$ -sparse vector and assume we are given noisy data  $\mathbf{Y} = \Psi\mathbf{H} + \xi$  with  $\|\xi\|_2 \leq \eta$ . Then the solution  $\hat{\mathbf{H}}$  to the BP problem satisfies  $\|\hat{\mathbf{H}} - \mathbf{H}\|_2 \leq C\eta$ . The constant  $C$  depends only on  $\delta_{3S}$  and  $\delta_{4S}$ . If  $\delta_{4S} \leq 1/3$  then  $C \leq 15.41$ . In particular, if no noise is present, i.e.,  $\eta = 0$ , then under the stated condition, BP recovers  $\mathbf{H}$  exactly.

### 3.2. The Measurement Matrix $\mathbf{A}$

For our application, we choose a measurement matrix  $\mathbf{A}$  in  $\Psi = \mathbf{A}\Phi$  which satisfies the *concentration inequality*

$$\mathbb{P}(|\|\mathbf{A}\mathbf{v}\|^2 - \|\mathbf{v}\|^2| \geq \varepsilon\|\mathbf{v}\|) \leq 2e^{-c\frac{\varepsilon^2}{2}}, \varepsilon \in (0, \frac{1}{3}),$$

for all  $\mathbf{v} \in \mathbb{R}^D$  and some constant  $c > 0$  [8], where  $\mathbb{P}(\cdot)$  denotes probability. This inequality is satisfied by the Gaussian ensemble random matrix [8]. Specifically, if the entries of  $\mathbf{A}$  are independent normal variables with mean zero and variance  $L^{-1}$ , then the *concentration inequality* holds with  $c = 7/18$ .

### 3.3. Stable Recovery Condition Using Basis Pursuit

In [8], the isometry constants of matrix  $\Psi = \mathbf{A}\Phi$  were estimated using the following theorem.

**Theorem 3.2** Let  $\Psi \in \mathbb{R}^{D \times K}$  be a dictionary with coherence  $\mu$ . Assume that  $S - 1 \leq \frac{1}{16}\mu^{-1}$ . Let  $\mathbf{A} \in \mathbb{R}^{L \times D}$  be a random matrix that satisfies the *concentration inequality*. Assume that

$$L \geq C_1(S \log(K/S) + C_2 + t), \quad (7)$$

where  $t$  is a positive real number. Then with probability at least  $1 - e^{-t}$ , the composed matrix  $\Psi = \mathbf{A}\Phi$  has restricted isometry constant  $\delta_S(\Psi) \leq 1/3$ . The constants satisfy  $C_1 \leq 138.51e^{-1}$  and  $C_2 \leq \log(1250/13) + 1 \approx 5.57$ .

Assuming that each column of matrix  $\Phi$  is normalized to 1, the coherence  $\mu$  of the matrix  $\Phi$  is defined as

$$\mu := \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|,$$

where  $\varphi_i$  and  $\varphi_j$  are the  $i$ th and  $j$ th column of  $\Phi$ , respectively. Combining Theorem 3.1 and Theorem 3.2, we can obtain the stable recovery condition for the spreading function vector  $\mathbf{H}$  using BP. Specifically, the number of necessary measurements  $L$  is on the order of  $S \log(K/S)$ .

## 4. WAVEFORM DESIGN

According to Theorem 3.2, a very low coherence of  $\Phi$  is required to satisfy the condition  $S - 1 \leq \frac{1}{16}\mu$  for compressive sampling. Also the research in [8] shows that if the columns of  $\Phi$  are orthogonal, the upper bound of the required number

of samples for stable recovery decreases. As a result, we need to design the desired transmission waveforms to generate the orthogonal basis in  $\Phi$  for different types of LTV systems.

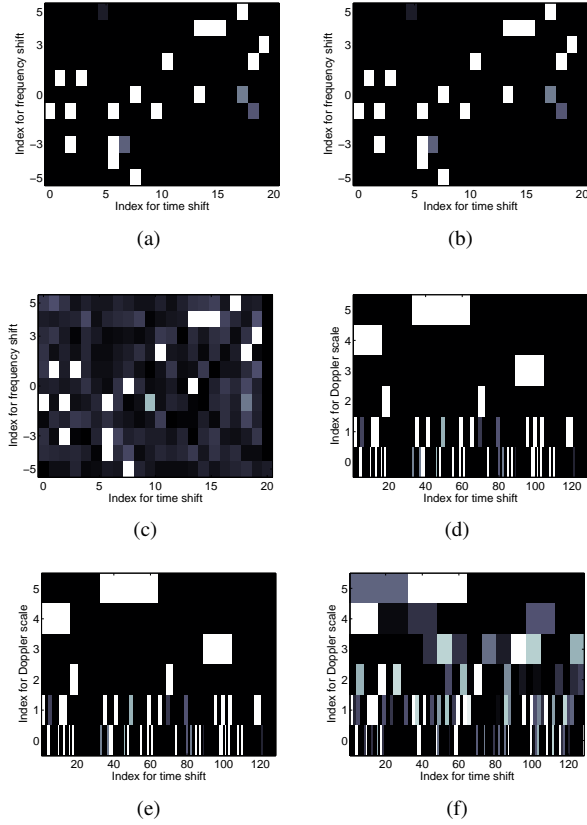
For narrowband systems, we can consider the direct sequence code division multiple access (DS-CDMA) signal as the basic transmission waveform. The DS-CDMA signal is generated using a pseudo noise (PN) sequence. If the length of the PN code is  $N_c$ , the waveform is designed as  $x(t) = \sum_{p=0}^{N_c-1} c_p v(t - pT_c)$  where  $c_p$  is the  $p$ th bit or chip of the PN sequence, and  $v(t)$  is the PN chip waveform with duration  $T_c$ . It is known that orthogonality exists between different TF shifted versions of these waveforms [1]. Specifically,  $\langle x_{m,n}, x_{m',n'} \rangle = \int_0^T x_{m,n}(t) x_{m',n'}^*(t) dt \approx C\delta[m - m']\delta[l - l']$  where  $\delta[\cdot]$  denotes the Kronecker delta function,  $C$  is a constant, and  $T = N_c T_c$  is the duration of the DS-CDMA signal waveform.

For wideband systems, in order to obtain a set of orthogonal basis, and due to the similarity of  $x_{n,m}(t)$  in (4) to orthonormal wavelet functions, we propose a wavelet-based waveform design scheme [2]. Specifically, if we let  $\psi(t)$  be a basis wavelet function and  $\psi_{n,m}(t) = 2^{-\frac{m}{2}} \psi(\frac{t}{2^m} - n)$ ,  $n, m \in \mathbb{Z}$  constitute an orthonormal basis on  $L^2(\mathbb{R})$ , then we can choose the signaling waveform in (4) to be  $x(t) = \frac{1}{\sqrt{T_w}} \psi(\frac{t}{T_w})$ , where  $T_w$  is some positive real number. Assuming that  $W \approx 1/T_w$  and letting  $a_0 = \frac{1}{2}$ , the signal  $x_{n,m}(t)$  in (4) also constitutes an orthonormal basis since  $\forall n, n', m, m' \in \mathbb{Z}$ , we have  $\frac{1}{T_w} \int_{-\infty}^{\infty} \psi_{m,n}(\frac{t}{T_w}) \psi_{m',n'}^*(\frac{t}{T_w}) dt = \delta[n - n'] \delta[m - m']$ .

For a dispersive system, as it is a unitarily equivalent representation to the narrowband system, we can use the corresponding warped version of the aforementioned narrowband waveform design scheme. We can design the waveform as  $x(t) = \sum_{p=0}^{N_c-1} c_p (\mathbf{U}_\xi^{-1} v)(t - pT_c)$ , which is also a form of DS-CDMA signaling. Using the unitary relationship, it can be shown that the corresponding basis functions  $x_{m,n}(t)$  are orthogonal to each other. As a result,  $\Phi$  will have orthogonal columns for the application of compressive sensing.

## 5. SIMULATIONS

We demonstrate the identification results for narrowband systems using numerical simulations. The parameters we use for the simulations are as follows. We design the transmission waveform using a PN code with length 1023, and we assume that the system causes  $M = 21$  time shifts and  $N = 11$  frequency shifts. Specifically, for  $x_{m,n}(t)$  in (2),  $m = 0, 1, \dots, 20$  and  $n = -5, -4, \dots, 5$ , respectively. Hence, the number of basis is  $K = MN = 231$  and the length of the output vector is  $D = 1043$  samples. We assume that the vector for the spreading function  $\mathbf{H}$  is  $S$ -sparse with  $S = \lfloor K/10 \rfloor = 23$ , where  $\lfloor \cdot \rfloor$  means rounding to the nearest smaller integer. We use a random Gaussian ensemble matrix for  $\mathbf{A}$ . The application of compressive sensing yields an  $\mathbf{A}$  matrix with dimension  $L \times D$  where  $L = \lfloor 4.5S \log K/S \rfloor = 238$ .



**Fig. 1.** (a) SF of a narrowband system. (b) Recovered SF using compressive sensing. (c) Recovered SF using compressive sensing with insufficient samples. (d) WSF of a wideband system. (e) Recovered WSF using compressive sensing. (f) Recovered WSF using compressive sensing with insufficient samples.

We denote the estimated spreading function as  $\hat{\mathbf{H}}$ , and define the estimation error rate as  $\epsilon = \mathbb{E} \left( \frac{\|\hat{\mathbf{H}} - \mathbf{H}\|_{\ell_1}}{\|\mathbf{H}\|_{\ell_1}} \right)$ , where  $\mathbb{E}(\cdot)$  denotes expectation. In the following simulations, we obtain  $\epsilon$  by averaging the results of 100 simulations. In Figures 1(a) and 1(b), we demonstrate the narrowband SF and its estimation. As we can see, the  $S$ -sparse vector  $\mathbf{H}$  can be perfectly recovered by 238 measurements (out of the  $D = 1043$  possible measurements) with  $\epsilon \approx 4.35 \times 10^{-6}$ . In Fig.1(c), we demonstrate an unsuccessful SF estimation using  $L = \lfloor 3.5S \log K/S \rfloor = 185$  samples, and we can see the evident estimation error in the recovered SF with  $\epsilon \approx 0.93$ .

We also investigate the system identification results for a wideband system. We design the transmission waveform using Daubechies wavelets, and we assume that there are  $M = 6$  Doppler-scale changes. Specifically, in (4),  $m = 0, 1, \dots, 5$ , and for each  $m$ , there exists  $N = \lfloor DT_s / (2^m T_w) \rfloor$  time shifts, where  $\lfloor 2^m T_w / T_s \rfloor$  is the length of the wavelet after scaling and  $T_s$  is the sampling period of the wavelet. The number of signal basis is  $K = 252$  and the length of the out-

put vector is  $D = 1024$  samples. We assume that the vector for the WSF  $\mathbf{H}$  is  $S$ -sparse with  $S = \lfloor K/5 \rfloor = 50$ , and we use a random Gaussian ensemble matrix for  $\mathbf{A}$ . The true WSF is shown in Fig. 1(d). The accurate estimation results are obtained using  $L = \lfloor 3S \log K/S \rfloor = 242$ . In Fig. 1(e), As we can see, the  $S$ -sparse vector  $\mathbf{H}$  was perfectly recovered using 242 measurements with  $\epsilon \approx 5.67 \times 10^{-4}$ . In Fig.1(f), we estimate the spreading function using  $L = \lfloor 1.5S \log K/S \rfloor = 121$  samples; the recovered WSF has a large error of  $\epsilon \approx 0.57$ .

## 6. CONCLUSION

In this paper, we demonstrated the application of compressive sensing for estimating the spreading functions of different types of LTV systems. Specifically, we investigated the stable recovery conditions for the spreading functions by designing the appropriate waveform for the different systems. Our simulations showed that we can estimate the LTV system using a reduced number of measurements. Note that we are currently considering noisy systems as this processing assumed a very high signal-to-noise ratio.

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