

RELATIONSHIPS BETWEEN RADAR AMBIGUITY AND CODING THEORY

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ABSTRACT

We investigate the theory of the finite discrete Heisenberg-Weyl group in relation to the development of adaptive radar. We contend that this group can form the basis for the representation of the radar environment in terms of operators on the space of waveforms. We also demonstrate, following recent developments in the theory of error correcting codes, that the finite discrete Heisenberg-Weyl group provides a unified basis for the construction of useful waveforms/sequences for radar, communications and the theory of error correcting codes.

1. INTRODUCTION

Modern radars have the capacity to adaptively switch waveforms on a pulse to pulse basis and to retain coherence over many pulses, but these capacities are only just beginning to be exploited. If we are to fully exploit this waveform agility in both modern and future radars two important problems need to be addressed. The first is the representation of the environment as it pertains to the transmission of radar waveforms. This includes both targets and background clutter. The second important problem is to ensure that one has a sufficiently flexible set of waveforms to enable the choice a waveform which are optimized for a given situation.

It is the purpose of the present paper to show that both of these problems can be approached to a large extent within the same mathematical framework. That is, through the theory of the discrete Heisenberg-Weyl group [1]. It is well known that the continuous Heisenberg-Weyl group has application to the theory of radar, for example see [2], but the discrete version of this group has received little attention in radar. Since, in practical terms, the resolution of a radar is finite, by choosing a fine enough discretization in range and Doppler we can treat the radar perfectly well with the one dimensional discrete Heisenberg-Weyl group. This has a number of advantages, one of which is that the radar environment can be represented by a matrix acting on the space of waveforms.

The m -dimensional discrete Heisenberg-Weyl group provides a unifying framework for a number of important sequences significant in the construction of phase coded radar waveforms, in

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communications as spreading sequences, and in the theory of error correcting codes. Among the sequences which can be associated with the Heisenberg-Weyl group are the first and second order Reed-Muller codes, the Welty sequences [3], and the Kerdock and Preparata codes [4], which are non-linear binary error correcting codes containing more codewords for a given minimum distance than any linear code. The Kerdock codes are associated with decomposition of the Heisenberg-Weyl group into disjoint maximally commutative subgroups.

The paper is organised as follows. In Section 2 we introduce the theory of the discrete Heisenberg-Weyl group. We then briefly develop the theory of discrete radar and introduce the ambiguity function of a waveform in this framework. Section 4 provides the main contribution of the paper, it extends the theory developed in [4] for the extraspecial 2-group, which is the discrete Heisenberg-Weyl group of Section 2 corresponding to $p = 2$, to the other Heisenberg-Weyl groups considered. We develop this theory based on analysis of the ambiguity functions associated with the irreducible representation of the group. In Sections 5 and 6 we relate the general theory to the known case of the Kerdock codes [4] and to discrete radar. We find that the theory that leads to the Kerdock codes in the multi-dimensional ($p = 2$) Heisenberg-Weyl group leads to linear frequency modulated waveforms when applied to discrete radar.

2. THE DISCRETE HEISENBERG-WEYL GROUPS

We begin by defining a configuration space $A = \mathbb{Z}_p^m$ consisting of m -tuples of elements from the integers modulo p . In this paper we will take p to be a prime number. Under addition A forms an Abelian group. In radar theory the space A , with $m = 1$ would represent discrete ranges, while in discrete quantum mechanics the space A could represent possible discrete positions for a particle.

Define a Hilbert space \mathcal{H} , having orthonormal basis

$$\{|\mathbf{a}\rangle : \mathbf{a} \in A\}. \quad (1)$$

which we refer to as the Dirac basis. Note that we use the “bra-ket” notation for elements of the Hilbert space. An arbitrary element $|\phi\rangle \in \mathcal{H}$ can be expanded in this basis as

$$|\phi\rangle = \sum_{\mathbf{a} \in A} \langle \mathbf{a} | \phi \rangle |\mathbf{a}\rangle, \quad (2)$$

where $\langle \cdot | \cdot \rangle$ is the inner product on \mathcal{H} .

The dual group of A , denoted \hat{A} , is comprised of the homomorphisms from the group A onto the unit circle Π in \mathbb{C} . \hat{A} is also an Abelian group (under multiplication) and is (since A is finite) isomorphic to A . This isomorphism is made explicit through identifying each $\mathbf{b} \in A$ with a $\gamma_{\mathbf{b}} \in \hat{A}$, such that

$$\gamma_{\mathbf{b}}(\mathbf{a}) = \omega^{\mathbf{b} \cdot \mathbf{a}}, \quad (3)$$

for all $\mathbf{a} \in A$, where $\omega = \exp(2\pi i/p)$ is the p^{th} root of unity and \cdot denotes the usual dot product on \mathbb{Z}_p^m . We see from (3) that the elements of \hat{A} are just discrete sinusoids, or multi-dimensional versions of such. To each element of $\gamma_{\mathbf{b}} \in \hat{A}$ we can assign a vector in \mathcal{H} by

$$|\hat{\mathbf{b}}\rangle = \frac{i^{m/2}}{\sqrt{|A|}} \sum_{\mathbf{a} \in A} \omega^{\mathbf{b} \cdot \mathbf{a}} |\mathbf{a}\rangle. \quad (4)$$

The set $\{|\hat{\mathbf{a}}\rangle : \mathbf{a} \in A\}$ also forms an orthonormal basis for \mathcal{H} , which we refer to as the Fourier basis. We can define the unitary Fourier transform operator relating this orthonormal basis to (1) by

$$F = \frac{i^{m/2}}{\sqrt{|A|}} \sum_{\mathbf{a}, \mathbf{b} \in A} \omega^{\mathbf{b} \cdot \mathbf{a}} |\mathbf{a}\rangle \langle \mathbf{b}|, \quad (5)$$

where $|\mathbf{a}\rangle \langle \mathbf{b}|$ represents the cross projection operator on \mathcal{H} whose action on $|\phi\rangle \in \mathcal{H}$ is $|\mathbf{a}\rangle \langle \mathbf{b}| |\phi\rangle = \langle \mathbf{b} | \phi \rangle |\mathbf{a}\rangle$

We will denote the group $A \times \hat{A} \simeq A \times A$, which is a vector space since \mathbb{Z}_p is a field, by \overline{E} . We will refer to \overline{E} as the *phase space*.

On \mathcal{H} we define the unitary operators $\{D(\mathbf{a}, \mathbf{b}) : (\mathbf{a}, \mathbf{b}) \in \overline{E}\}$ by

$$D(\mathbf{a}, \mathbf{b}) = \sum_{\mathbf{c} \in A} \omega^{\mathbf{b} \cdot \mathbf{c}} |\mathbf{c} + \mathbf{a}\rangle \langle \mathbf{c}|. \quad (6)$$

Two such operators have the multiplication rule

$$D(\mathbf{a}, \mathbf{b})D(\mathbf{a}', \mathbf{b}') = \omega^{\mathbf{b} \cdot \mathbf{a}'} D(\mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}'), \quad (7)$$

from which we have the commutator

$$D(\mathbf{a}, \mathbf{b})^\dagger D(\mathbf{a}', \mathbf{b}') D(\mathbf{a}, \mathbf{b}) D(\mathbf{a}', \mathbf{b}')^\dagger = \omega^{\mathbf{a} \cdot \mathbf{b}' - \mathbf{a}' \cdot \mathbf{b}} I, \quad (8)$$

where † denotes adjoint.

The set of unitary operators on \mathcal{H}

$$E = \{T(\lambda, \mathbf{a}, \mathbf{b}) = \vartheta^\lambda D(\mathbf{a}, \mathbf{b}) : \lambda \in \mathbb{Z}_{2p}, (\mathbf{a}, \mathbf{b}) \in \overline{E}\}, \quad (9)$$

forms an representation of the discrete Heisenberg-Weyl group on \mathcal{H} . This representation is irreducible [1, 4]. This means that there are no nontrivial subspaces of \mathcal{H} invariant under the action of E .

Now the center of the group E , $Z(E)$, consists of the elements $\{\vartheta^\lambda I : \lambda \in \mathbb{Z}_{2p}\}$, where I is the identity operator on \mathcal{H} . The factor space $E/Z(E)$ is easily seen to be the phase space \overline{E} .

Considering the commutation relation (8) we can define the *symplectic* inner product

$$((\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')) = \mathbf{a} \cdot \mathbf{b}' - \mathbf{a}' \cdot \mathbf{b}, \quad (10)$$

on the phase space \overline{E} , and note that two operators $D(\mathbf{a}, \mathbf{b})$ and $D(\mathbf{a}', \mathbf{b}')$ commute if and only if $((\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')) = 0$. We may then identify subgroups of E consisting mutually commuting sets of operators $D(\mathbf{a}, \mathbf{b})$ with isotropic subspaces of \overline{E} . A subspace \overline{H}

of \overline{E} is isotropic if any pair of points $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}') \in \overline{H}$ satisfy $((\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')) = 0$. A isotropic subspace \overline{H} of \overline{E} corresponds to the Abelian subgroup $\{D(\mathbf{a}, \mathbf{b}) : (\mathbf{a}, \mathbf{b}) \in \overline{H}\}$ of E .

Finally, in this section, we consider the space of linear operators \mathcal{O} on the Hilbert space \mathcal{H} . We have the following theorem, which can be prove by substituting (6) into (11):

Theorem 1. Any operator $S \in \mathcal{O}$ can be represented as

$$S = \frac{1}{|A|} \sum_{(\mathbf{a}, \mathbf{b}) \in \overline{E}} \text{Tr}(D(\mathbf{a}, \mathbf{b})^\dagger S) D(\mathbf{a}, \mathbf{b}). \quad (11)$$

3. DISCRETE RADAR

Let us see how the above theory applies to radar. For radar the configuration space $A = \mathbb{Z}_p$ consists of a large number p of discrete times. To make the development more transparent we label the elements of A by $\tau \in \mathbb{Z}_p$ and \hat{A} by $\nu \in \mathbb{Z}_p$, rather than by \mathbf{a} and \mathbf{b} . ν/p is the digital frequency. The phase space \overline{E} in this case is the time-frequency plane. The vectors $|\phi\rangle \in \mathcal{H}$ are our waveforms, and their expansion coefficients in the Dirac basis $\phi(\tau) = \langle \tau | \phi \rangle$, give their p -periodic time sequences. The Dirac basis waveforms $|\tau\rangle$ correspond to impulses at time τ . The Fourier basis correspond to fixed frequency sinusoidal waveforms, since for these have coefficients $\langle \tau | \hat{\nu} \rangle = \sqrt{i/p} \omega^{\nu\tau}$.

Abstractly, the operation of the radar consists of transmitting a waveform $|\phi\rangle \in \mathcal{H}$, which is reflected by the environment, or radar scene, and returns as the waveform $|\psi\rangle \in \mathcal{H}$. Thus, the radar scene can be considered an operator, S , on \mathcal{H} . The expansion (11)

$$S = \sum_{(\tau, \nu) \in \overline{E}} \sigma(\tau, \nu) D(\tau, \nu), \quad (12)$$

can then be considered as a decomposition of the radar scene into point scatters, each of which delay the waveform by a time τ and Doppler shift the waveform by ν , with the return being multiplied by a complex scattering amplitude $\sigma(\tau, \nu)$. Theorem 1, implies that the scatterer distribution $\sigma(\tau, \nu) = \text{Tr}(D(\tau, \nu)^\dagger S)/|A|$.

Suppose that we have an unknown radar scene S and we would like to learn something about it. We transmit a waveform $|\phi\rangle$ and note the return $|\psi\rangle$. In the absence of noise we now know that $|\psi\rangle = S|\phi\rangle$, or that

$$S = |\psi\rangle \langle \phi| + R, \quad (13)$$

where the operator R , which annihilates $|\phi\rangle$, $R|\phi\rangle = 0$, is undetermined. Thus, as a result of transmitting $|\phi\rangle$, we now know the action of the operator $\tilde{S} = S|\phi\rangle \langle \phi|$, which in terms of scatterer distributions is

$$\tilde{S} = \sum_{(\tau, \nu) \in \overline{E}} \tilde{\sigma}(\tau, \nu) D(\tau, \nu), \quad (14)$$

where

$$\tilde{\sigma}(\tau, \nu) = \sum_{(\tau', \nu') \in \overline{E}} \sigma(\tau', \nu') \mathcal{A}_\phi(\tau' - \tau, \nu' - \nu) \omega^{\nu(\tau' - \tau)}. \quad (15)$$

and the *ambiguity function*, \mathcal{A} , given by

$$\mathcal{A}_\phi(\tau, \nu) = \text{Tr}(D(\tau, \nu) |\phi\rangle \langle \phi|) = \langle \phi | D(\tau, \nu) | \phi \rangle. \quad (16)$$

We now go back and consider the ambiguity function in the more general setting of Section 2.

4. COVARIANT TIGHT FRAMES AND AMBIGUITY FUNCTIONS

The ambiguity function of a normalised vector $|\phi\rangle \in \mathcal{H}$, $\mathcal{A}_\phi : \bar{E} \rightarrow \mathbb{C}$, is defined as

$$\mathcal{A}_\phi(\mathbf{a}, \mathbf{b}) = \langle \phi | D(\mathbf{a}, \mathbf{b}) | \phi \rangle, \quad (17)$$

that is, the inner product of $|\phi\rangle$ with $D(\mathbf{a}, \mathbf{b})|\phi\rangle$. An important property of ambiguity functions is Moyal's identity

$$\frac{1}{|A|} \sum_{(\mathbf{a}, \mathbf{b}) \in \bar{E}} \overline{\mathcal{A}_\phi(\mathbf{a}, \mathbf{b})} \mathcal{A}_\psi(\mathbf{a}, \mathbf{b}) = |\langle \phi | \psi \rangle|^2, \quad (18)$$

This result is obtained by expanding the projector $|\phi\rangle\langle\phi|$ according to (11), multiplying by the projector $|\psi\rangle\langle\psi|$ and taking the trace of the resulting equation. A special case of Moyal's identity is

$$\frac{1}{|A|} \sum_{(\mathbf{a}, \mathbf{b}) \in \bar{E}} |\mathcal{A}_\phi(\mathbf{a}, \mathbf{b})|^2 = 1. \quad (19)$$

We can understand a great deal about the structure ambiguity functions associated with various vectors \mathcal{H} , by understanding the orbits in \mathcal{H} under the action of the Heisenberg-Weyl group E . The orbit containing the vector $|\phi\rangle$ consists of the set of vectors

$$\{|\lambda, \mathbf{a}, \mathbf{b}, \phi\rangle = T(\lambda, \mathbf{a}, \mathbf{b})|\phi\rangle : \lambda \in \mathbb{Z}_{2p}, (\mathbf{a}, \mathbf{b}) \in \bar{E}\}. \quad (20)$$

Such orbits are called *coherent states* in the physics literature [5], and as we demonstrate below they form tight frames [6] of vectors in \mathcal{H} . Here we shall refer to these as covariant tight frames (CTF) and to $|\phi\rangle$ as the fiducial vector of the CTF.

Of importance in understanding the structure of the orbit (20) is the isotropy subgroup of the fiducial vector $|\phi\rangle$. The isotropy subgroup of $|\phi\rangle$ consists of those T which merely multiply $|\phi\rangle$ by a phase,

$$T(\lambda, \mathbf{a}, \mathbf{b})|\phi\rangle = e^{i\chi(\lambda, \mathbf{a}, \mathbf{b})}|\phi\rangle. \quad (21)$$

Obviously, the isotropy subgroup H_ϕ of $|\phi\rangle$ is at least $Z(E)$, the centre of E , although it may be larger. If the isotropy subgroup is $Z(E)$, then the orbit will be parameterized by the phase space \bar{E} . The isotropy subspace of $|\phi\rangle$ is $\bar{H}_\phi = H_\phi/Z(E) \subset \bar{E}$ and the orbit is parameterized by the cosets $\mathcal{C}_\phi = \bar{E}/\bar{H}_\phi$.

Thus, given a fiducial vector $|\phi\rangle$, we consider the set of vectors

$$\{|\mathbf{a}, \mathbf{b}, \phi\rangle = D(\mathbf{a}, \mathbf{b})|\phi\rangle : (\mathbf{a}, \mathbf{b}) \in \mathcal{C}_\phi\}. \quad (22)$$

We have the following theorems, which will be proved elsewhere:

Theorem 2. *Let $|\phi\rangle \in \mathcal{H}$ be a normalized vector with isotropy subspace $\bar{H}_\phi \subset \bar{E}$. Then, either*

1. \bar{H}_ϕ is a maximal isotropic subspace of \bar{E} , \mathcal{A}_ϕ is unimodular on \bar{H}_ϕ and zero on \bar{H}_ϕ^c , and the CTF $\{|\mathbf{a}, \mathbf{b}, \phi\rangle = D(\mathbf{a}, \mathbf{b})|\phi\rangle : (\mathbf{a}, \mathbf{b}) \in \mathcal{C}_\phi\}$ is an orthonormal basis, or
2. $\bar{H}_\phi = \{(\mathbf{0}, \mathbf{0})\}$ and the corresponding CTF is parameterized by the entire phase space \bar{E} .

We will say the two maximal isotropic subspaces \bar{H}_ϕ and \bar{H}_ψ are disjoint if $\bar{H}_\phi \cap \bar{H}_\psi = \{(\mathbf{0}, \mathbf{0})\}$.

As a consequence of Moyal's identity we also have the following theorem which relates to the "angle" between the orthonormal bases associated with two fiducial vector having disjoint maximal isotropic isotropy subspaces.

Theorem 3. *Let $|\phi\rangle$ and $|\psi\rangle \in \mathcal{H}$ have maximal isotropic isotropy subspaces \bar{H}_ϕ and \bar{H}_ψ which are disjoint, then*

$$|\langle \mathbf{a}, \mathbf{b}, \phi | \mathbf{a}', \mathbf{b}', \psi \rangle| = \frac{1}{\sqrt{|A|}}, \quad (23)$$

for all $(\mathbf{a}, \mathbf{b}) \in \mathcal{C}_\phi$ and $(\mathbf{a}', \mathbf{b}') \in \mathcal{C}_\psi$.

An example of two disjoint maximal isotropic subspaces is $\bar{H}_D = \{(\mathbf{0}, \mathbf{b}) : \mathbf{b} \in A\}$ and $\bar{H}_F = \{(\mathbf{a}, \mathbf{0}) : \mathbf{a} \in A\}$. \bar{H}_D has the CTF given by the orthonormal basis (1) and can be associated with fiducial vector $|\mathbf{0}\rangle$, while \bar{H}_F has the CTF given by the orthonormal basis (4) with fiducial vector $|\hat{\mathbf{0}}\rangle$. The Fourier transform operator F given in (5) transforms $D(\mathbf{a}, \mathbf{b})$ as

$$F^\dagger D(\mathbf{a}, \mathbf{b}) F = \omega^{-\mathbf{a} \cdot \mathbf{b}} D(\mathbf{b}, -\mathbf{a}), \quad (24)$$

Thus F induces a symplectic action, i.e., it preserves the inner product (10), on the phase space given by $f : \bar{E} \rightarrow \bar{E}$, such that $f(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, -\mathbf{a})$. This action exchanges \bar{H}_D and \bar{H}_F .

The question arises as to whether it is possible to choose a set of vectors (waveforms) such that the supports of their ambiguity functions are non-intersecting (except at $(\mathbf{0}, \mathbf{0})$), while jointly covering the whole of phase space. This is equivalent to covering the whole of the phase space \bar{E} with disjoint maximal isotropic subspaces. At least in certain instances the answer is yes. The construction in these cases works as follows [4].

Define a symplectic transformation on \mathcal{H} by

$$W(P) = \sum_{\mathbf{c} \in A} \bar{\vartheta}^{\mathbf{c} \cdot P \mathbf{c}} |\mathbf{c}\rangle\langle\mathbf{c}|, \quad (25)$$

where P is a symmetric matrix on \mathbb{Z}_p . It is important to note that in this definition the quadratic form $\mathbf{c} \cdot P \mathbf{c}$ is to be calculated in \mathbb{Z}_{2p} . We have

$$W(P)^\dagger D(\mathbf{a}, \mathbf{b}) W(P) = \vartheta^{\mathbf{a} \cdot P \mathbf{a}} D(\mathbf{a}, \mathbf{b} + P\mathbf{a}). \quad (26)$$

$W(P)$ induces an action on the phase space $w_P : \bar{E} \rightarrow \bar{E}$, such that $w_P(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{b} + P\mathbf{a})$. This action preserves the symplectic inner product (10), since P is symmetric, and so it maps maximal isotropic subspaces to other such subspaces. The scheme is then to generate new maximal isotropic subspaces \bar{H}_P as

$$\bar{H}_P = w_P(\bar{H}_F) = \{(\mathbf{a}, P\mathbf{a}) : \mathbf{a} \in A\}. \quad (27)$$

Two such subspaces corresponding to symmetric matrices P and Q will intersect at only at solutions of $(P - Q)\mathbf{a} = \mathbf{0}$. Thus, the problem of covering \bar{E} with disjoint maximal isotropic subspaces will be solved if we can find a set of $|A| - 1$ non-singular symmetric matrices \mathcal{P} , over \mathbb{Z}_p , such that for any pair of matrices $P, Q \in \mathcal{P}$, $P - Q$ is non-singular. The covering would then be

$$\bar{E} = \bar{H}_D \cup \bar{H}_F \cup \left(\bigcup_{P \in \mathcal{P}} w_P(\bar{H}_F) \right). \quad (28)$$

When this occurs the set of vectors

$$\{|\mathbf{a}\rangle : \mathbf{a} \in A\} \cup \{|\hat{\mathbf{b}}\rangle : \mathbf{b} \in A\} \cup \{i^\lambda W(P)|\hat{\mathbf{b}}\rangle : \lambda \in \mathbb{Z}_4, \mathbf{b} \in A, P \in \mathcal{P}\}, \quad (29)$$

is such that the magnitude of the inner product of any pair of vector in the set is either 0 or $1/\sqrt{|A|}$. We discuss two cases in which such a covering can be constructed.

5. \mathbb{Z}_4 -KERDOCK CODES

Kerdock codes [7] are non-linear binary error correcting codes which contain more codewords for a given minimum distance than any linear code. It was shown by Hammons et al. [8] that the Kerdock codes could be constructed as binary images under the Gray map of linear codes over \mathbb{Z}_4 . The geometry of these codes was studied extensively by Calderbank et al. [4] who demonstrated their relationship to the *extraspecial 2-group*. This group is identical to the discrete Heisenberg-Weyl group (9) for $p = 2$, and most of the theory developed in Section 4 can be found in [4] for this case.

Here the configuration space $A = \mathbb{Z}_2^m$ consist of the binary sequences of length m . This case has been studied extensively in the theory of error correction codes [4]. The Fourier basis is

$$|\hat{\mathbf{b}}\rangle = \left(\frac{i}{2}\right)^{m/2} \sum_{(\mathbf{a}, \mathbf{b}) \in \bar{E}} (-1)^{\mathbf{b} \cdot \mathbf{a}} |\mathbf{a}\rangle. \quad (30)$$

Apart from the normalizing constant $(i/2)^{m/2}$, the coefficients of the Fourier basis are related to the the first order Reed-Muller code $RM(1, m+1)$, in the following sense. If we apply group E , from (9), to the vector $|\hat{\mathbf{0}}\rangle$ we obtain the set of vectors

$$\{i^\lambda |\hat{\mathbf{b}}\rangle : \lambda \in \mathbb{Z}_4, \mathbf{b} \in A\}, \quad (31)$$

In the Dirac basis, neglecting the common normalization factor $(i/2)^{m/2}$, the coefficients of these vectors form $RM(1, m+1)$ as a linear code of length 2^m over \mathbb{Z}_4 . If we then apply the Gray map, $\{1 \rightarrow 00, i \rightarrow 01, (-1) \rightarrow 11, (-i) \rightarrow 10\}$, we then obtain the conventional form of $RM(1, m+1)$ as a binary code of length 2^{m+1} . In a similar way the second order Reed-Muller code $RM(2, m+1)$ corresponds to the set of vectors

$$\{i^\lambda W(P) |\hat{\mathbf{b}}\rangle : \lambda \in \mathbb{Z}_4, \mathbf{b} \in A, P \text{ a binary symmetric matrix}\}. \quad (32)$$

In this case many are many possible sets of binary symmetric matrices \mathcal{P} which lead to a disjoint covering of the phase space with maximal isotropic subspaces [4]. One possibility consists of a vector space of non-singular Hankel matrices, with one binary symmetric matrix with any given diagonal. The sets of the form

$$\{i^\lambda W(P) |\hat{\mathbf{b}}\rangle : \lambda \in \mathbb{Z}_4, \mathbf{b} \in A, P \in \mathcal{P}\} \cup \{|\mathbf{a}\rangle, \mathbf{a} \in A\}, \quad (33)$$

are the Kerdock codes. Note that here the zero matrix is in \mathcal{P} and that $W(0) = I$. Obviously, the first set above lies within the second order Reed-Muller code (32). The Welti sequences correspond to a particular choice of P in (33).

6. DISCRETE RADAR REVISITED

In this case, since $m = 1$, the matrices in \mathcal{P} are just numbers. In fact, since p is prime, we can take $\mathcal{P} = \{0, 1, \dots, p-1\}$. The $p+1$ maximal isotropic subspaces consist of the line $\{(0, \tau) : \tau \in \{0, \dots, p-1\}\}$ along with the lines $\{(\tau, n\tau) : \tau \in \{0, \dots, p-1\}\}$, for $n \in \mathcal{P}$, in the phase space or time-frequency plane. This time-frequency plane covering is displayed in Table 6 for $p = 11$. The corresponding vectors (waveforms) are, for $n \in \mathcal{P}$,

$$|n, \hat{\nu}\rangle = W(n) |\hat{\nu}\rangle = \sqrt{\frac{i}{p}} \sum_{\tau=0}^{p-1} \theta^{n\tau^2} \omega^{\nu\tau} |\tau\rangle, \quad (34)$$

Table 1. Tiling of the time-frequency plane by maximal isotropic subspaces for $p = 11$.

0	11	6	8	9	3	10	4	5	7	2	
0	10	11	4	6	5	8	7	9	2	3	
0	9	5	11	3	7	6	10	2	8	4	
0	8	10	7	11	9	4	2	6	3	5	
0	7	4	3	8	11	2	5	10	9	6	
ν	0	6	9	10	5	2	11	8	3	4	7
0	5	3	6	2	4	9	11	7	10	8	
0	4	8	2	10	6	7	3	11	5	9	
0	3	2	9	7	8	5	6	4	11	10	
0	2	7	5	4	10	3	9	8	6	11	
\star	1	1	1	1	1	1	1	1	1	1	
						τ					

for $\nu \in \{0, \dots, p-1\}$. These are linear frequency modulated sinusoid or chips. The ambiguity function of such a chirped waveform $|n, \hat{\nu}'\rangle$ is

$$\mathcal{A}_{n, \hat{\nu}'}(\tau, \nu) = \theta^{n\tau^2} \omega^{\tau\nu'} \mathcal{A}_{\hat{\mathbf{0}}}(\tau, \nu + n\tau), \quad (35)$$

where $\mathcal{A}_{\hat{\mathbf{0}}}$ is the ambiguity function of the waveform $|\hat{\mathbf{0}}\rangle$. Thus, the magnitude of the ambiguity (35) is

$$|\mathcal{A}_{n, \hat{\nu}'}(\tau, \nu)| = |\mathcal{A}_{\hat{\mathbf{0}}}(\tau, \nu + n\tau)| = \delta_{\nu+n\tau, 0}. \quad (36)$$

Thus, we can explicitly see the how the waveforms $\{|n, \hat{\nu}\rangle : n \in \mathcal{P}, \nu \in \{0, \dots, p-1\}\}$ along with the Dirac basis waveform $\{|\tau\rangle : \tau \in \{0, \dots, p-1\}\}$, have ambiguity functions with disjointly cover the time-frequency plane.

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