$$\begin{aligned}\frac{\mathcal{F}^{(D)}(k)}{L} &= (D-k)\log\left(\frac{1}{D-k}\sum_{i=k+1}^{D}\hat{\rho}_i\right) - \sum_{i=k+1}^{D}\log\hat{\rho}_i \\ &= (D-k)\log\left(\sigma_n^2\left(1+\frac{\sum_{i=k+1}^{D}\left(\hat{\rho}_i-\sigma_n^2\right)}{(D-k)\,\sigma_n^2}\right)\right) - \sum_{i=k+1}^{D}\log\left(\sigma_n^2\left(1+\frac{\hat{\rho}_i-\sigma_n^2}{\sigma_n^2}\right)\right) \\ &= (D-k)\left(\frac{\sum_{i=k+1}^{D}\left(\hat{\rho}_i-\sigma_n^2\right)}{(D-k)\,\sigma_n^2} - \frac{1}{2}\left(\frac{\sum_{i=k+1}^{D}\left(\hat{\rho}_i-\sigma_n^2\right)}{(D-k)\,\sigma_n^2}\right)^2 + O\left(\left(\frac{\log\log L}{L}\right)^{(3/2)}\right)\right) \\ &\quad - \sum_{i=k+1}^{D}\left(\frac{\hat{\rho}_i-\sigma_n^2}{\sigma_n^2} - \frac{1}{2}\left(\frac{\hat{\rho}_i-\sigma_n^2}{\sigma_n^2}\right)^2 + O\left(\left(\frac{\log\log L}{L}\right)^{(3/2)}\right)\right) \\ &= \frac{1}{2}\left(\sum_{i=k+1}^{D}\left(\frac{\hat{\rho}_i-\sigma_n^2}{\sigma_n^2}\right)^2 - \frac{1}{D-k}\left(\sum_{i=k+1}^{D}\frac{\hat{\rho}_i-\sigma_n^2}{\sigma_n^2}\right)^2\right) + O\left(\left(\frac{\log\log L}{L}\right)^{(3/2)}\right), \end{aligned} \tag{E7}$$

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Joint Transmitter and Receiver Polarization Optimization for Scattering Estimation in Clutter

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***Abstract*—Controlling the polarization information in transmitted waveforms enables improving the performance of radar systems. We consider the design of optimal polarizations at both the radar transmitter and receiver for the estimation of target scattering embedded in clutter. The goal is to minimize the mean squared error of the scattering estimation subject to an average radar pulse power constraint. Under the condition that the target and clutter scattering covariance matrices are known a priori, we show that such a problem is equivalent to the optimal design of a radar sensing matrix that contains the polarization information. We formulate the optimal design as a nonlinear optimization problem and then recast it in a convex form and is thus efficiently solvable by semi-definite programming (SDP). We compare the sensing performance of the optimally selected polarization over conventional approaches. Our numerical results demonstrate that a significant amount of power gain is achieved in the target scattering estimation through such an optimal design.**

***Index Terms*—Adaptive estimation, optimization methods, radar polarimetry, scattering matrices.**

I. INTRODUCTION

Advances in digital signal processing and computing technology have resulted in the emergence of increasingly adaptive radar systems.

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It is clear that adaptive radar systems have more robust performance by adapting their sensing patterns (or more specifically waveforms) to the operation scenarios including target, clutter etc. Adaptive waveform design has attracted a lot of attention recently [1]–[10]. In this correspondence we design the optimal waveform polarization for the estimation of target scattering in an environment with clutter (e.g., when the target is close to a sea surface). We consider the optimal transmitter and receiver polarization for a polarimetric radar by adapting to the target and clutter polarimetric characteristics for enhanced sensing performance.

Polarimetric information of the radar targets reveals target details such as geometrical structure, shape, reflectivity, and orientation. Radar polarimetry can be used not only for target classification but also for enhancing target detection and estimation performance, i.e., resolving targets in a clutter environment [8], [11]. To obtain the target polarimetric scattering information, conventional polarimetric radar systems which transmitter waveforms with both horizontal (H) and vertical (V) orientations have been developed and adopted in various applications [12]. Such systems alternately switch between the two orthogonal polarizations at both the transmitter and receiver sides, and thus result in four combinations of transmitter and receiver polarizations: HH, HV, VH, and VV.

In modern radar systems, any polarization on either transmission or reception can be synthesized by using the linear combinations of the H and V components. Thus, besides the four types of transmitter/receive combinations above, such radar can achieve any pair of transmitter/recieve polarizations. Such flexibility greatly enhances the polarimetric sensing capability of the radar system. An example, the exploration of adaptive polarization for polarimetric contrast enhancement has been widely studied in the synthetic aperture radar imaging [13], [14].

In this correspondence, we consider the radar waveform polarization optimization and power scheduling in the estimation of target scattering in clutter. Under the condition that the target and clutter scattering covariance matrices are known *a priori*, we cast such a problem as the optimal design of the radar sensing matrix that is determined by the radar transmitter/receiver polarization and waveform power levels. The optimal design of the sensing matrix for a linear Gaussian model was pursued in [15], which has an analytical solution and can be considered as a special case of our model when clutter is not present. Due to the coupling of the clutter with the transmitter waveforms, the resulted optimization problem is highly nonlinear but can be recast as a semi-definite programming (SDP). This enables an efficient numerical solution which demonstrate that the optimally selected polarization has a few dB power gain over the conventional fixed polarization approaches.

We also compare the performance of the scattering estimation achieved by joint transmitter/receiver optimization with the cases when there is optimization at either the transmitter or receiver only. We demonstrate that optimization at the transmitter can bring larger performance improvement than the receiver due to fact that the optimization at the transmitter enables the optimal options of not only waveform polarizations but also power levels among pulses.

Notation: A lower case letter (e.g., a) denotes a scalar, a boldface/lowercase letter (e.g., $\mathbf{a}$) denotes a vector, and a boldface/uppercase letter (e.g., $\mathbf{A}$) denotes a matrix. In addition, $\mathrm{Tr}(\mathbf{A})$, $\mathbf{A}^T$, and $\mathbf{A}^H$ denote the trace, transpose, and Hermitian of $\mathbf{A}$ respectively. The letter $\mathbf{I}_n$ denotes an identity matrix of size $n \times n$. For two matrices $\mathbf{B}$ and $\mathbf{C}$, the relation $\mathbf{B} \succeq \mathbf{C}$ means that $\mathbf{B} - \mathbf{C}$ is positive semi-definite.

II. Problem Formulation

Consider a polarized waveform transmitted from a radar transmitter. When the transmitted waveform encounters a target or the clutter in the far field, another field is returned and received by the radar receiver. The two electric fields are related to each other by means of the target or clutter scattering matrices [16].

Specifically we assume that the radar transmits a polarized waveform

$$\mathbf{s}(t) = \sqrt{P}\boldsymbol{\xi} s(t) = [\xi_h, \xi_v]^H s(t)$$

where $\boldsymbol{\xi}$ is the transmitter polarization vector, $s(t)$ is the pulse shape, and P is the transmitter power. It is assumed that $\|\boldsymbol{\xi}\| = 1$ and the pulse has unitary energy, i.e., $\|s^2\| = \int_{-\infty}^{\infty} |s(t)|^2 dt = 1$. In addition, we assume that the receiver antenna has polarization $\boldsymbol{\eta} = [\eta_h, \eta_v]^H$ with $\|\boldsymbol{\eta}\| = 1$. After ignoring the target Doppler shift, we obtain that the complex envelope of the received signal at the radar receiver can be represented as [8], [17]

$$y(t) = \frac{g}{r^2}\sqrt{P}\boldsymbol{\eta}^H(\mathbf{S}_t + \mathbf{S}_c)\boldsymbol{\xi}\, s(t-\tau) + w(t). \tag{1}$$

In the above equation, $\mathbf{S}_{\mathrm{t}}$ and $\mathbf{S}_{\mathrm{c}}$ are the target and clutter scattering matrices respectively, $w(t)$ is the white noise process, r is the distance from the target to the radar, τ is the delay resulted from waveform forward and backward propagation, and g is a constant depending on the radar system characteristics such as operating frequency, permittivity and permeability of free space, and antenna gain at the target illumination angle etc.

After performing a matched filtering on (1) and a normalization by absorbing the constant g/r^2 into $v(t)$, we obtain the following observation model:

$$y = \sqrt{P}\,\boldsymbol{\eta}^H(\mathbf{S}_t + \mathbf{S}_c)\boldsymbol{\xi} + v \tag{2}$$

where v is white noise with variance σ_v^2. The scattering matrices $\mathbf{S}_{\mathrm{t}}$ and $\mathbf{S}_{\mathrm{c}}$ are represented by 2×2 S-matrices [17], [8], which describe completely the polarization transforming properties of the target and clutter. We assume that they have the following matrix representation

$$\mathbf{S}_{\mathrm{t}} = \begin{bmatrix} s_{\mathrm{hh}}^t & s_{\mathrm{hv}}^t \\ s_{\mathrm{vh}}^t & s_{\mathrm{vv}}^t \end{bmatrix}, \quad \mathbf{S}_{\mathrm{c}} = \begin{bmatrix} s_{\mathrm{hh}}^c & s_{\mathrm{hv}}^c \\ s_{\mathrm{vh}}^c & s_{\mathrm{vv}}^c \end{bmatrix}. \tag{3}$$

Our goal is to estimate $\mathbf{S}_{\mathrm{t}}$ based on radar measurement y which includes returns from both the target and the clutter, which is further degraded by thermal and background noise. For notational convenience, we convert (2) into a linear observation model by vectorizing $\mathbf{S}_{\mathrm{t}}$ and $\mathbf{S}_{\mathrm{t}}$. Specifically, we introduce

$$\mathbf{x}_{\mathrm{t}} = \begin{bmatrix} s_{\mathrm{hh}}^t & s_{\mathrm{vv}}^t & s_{\mathrm{hv}}^t & s_{\mathrm{vh}}^t \end{bmatrix}^T$$
$$\mathbf{x}_{\mathrm{c}} = \begin{bmatrix} s_{\mathrm{hh}}^c & s_{\mathrm{vv}}^c & s_{\mathrm{hv}}^c & s_{\mathrm{vh}}^c \end{bmatrix}^T$$

and

$$\mathbf{a}(P;\boldsymbol{\xi},\boldsymbol{\eta}) = \sqrt{P}\mathbf{p}(\boldsymbol{\xi},\boldsymbol{\eta}) \stackrel{\mathrm{def}}{=} \sqrt{P}[\xi_{\mathrm{h}}\eta_{\mathrm{h}} \quad \xi_{\mathrm{v}}\eta_{\mathrm{v}} \quad \xi_{\mathrm{h}}\eta_{\mathrm{v}} \quad \xi_{\mathrm{v}}\eta_{\mathrm{h}}]^T. \tag{4}$$

With the above notation, we can rewrite (2) as

$$y = \mathbf{a}(P;\boldsymbol{\xi},\boldsymbol{\eta})^T\mathbf{x}_{\mathrm{t}} + \mathbf{a}(P;\boldsymbol{\xi},\boldsymbol{\eta})^T\mathbf{x}_{\mathrm{c}} + v.$$

In addition, it is easy to see that

$$\|\mathbf{p}(\xi,\eta)\|^2 = \|\boldsymbol{\xi}\|^2\|\boldsymbol{\eta}\|^2 = 1.$$

To estimate the full polarimetric information (i.e., all component of $\mathbf{x}_{\mathrm{t}}$), multiple pulses of different polarizations need to be transmitted to obtain multiple measurements y. Suppose there are m pulses transmitted to measure $\mathbf{x}_{\rm t}$. We use $P(i), \boldsymbol{\xi}(i)$, and $\boldsymbol{\eta}(i)$ to denote the power, transmit, and receiver polarization of these pulses. The observation from these m pulses can thus be written as

$$\begin{aligned} y(i) &= \mathbf{a}(P(i);\boldsymbol{\xi}(i),\boldsymbol{\eta}(i))^T\mathbf{x}_{\rm t} \\ &\quad + \mathbf{a}(P(i);\boldsymbol{\xi}(i),\boldsymbol{\eta}(i))^T\mathbf{x}_{\rm c} \\ &\quad + v(i), \quad i = 1,2,\dots,m. \end{aligned} \tag{5}$$

We assumed that during the period of the m pulses, both the target scattering $\mathbf{x}_{\rm t}$ and the clutter scattering $\mathbf{x}_{\rm c}$ remain unchanged.

Introducing vector notations

$$\begin{aligned} \mathbf{y} &= [y_1, y_2, \dots, y_m]^T, \\ \mathbf{A} &= [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m]^T \quad \text{with} \\ \mathbf{a}_i &\stackrel{\text{def}}{=} \mathbf{a}(\xi(i), \eta(i)), \\ \mathbf{v} &= [v_1, v_2, \dots, v_m]^T \end{aligned} \tag{6}$$

we obtain a matrix representation of the m measurements in (5)

$$\mathbf{y} = \mathbf{A}\mathbf{x}_{\rm t} + \mathbf{A}\mathbf{x}_{\rm c} + \mathbf{v}. \tag{7}$$

For both $\mathbf{x}_{\rm t}$ and $\mathbf{x}_{\rm c}$, we further assume they both have a complex Gaussian distribution with known covariance matrix $\mathbf{C}_{\rm t}$ and $\mathbf{C}_{\rm c}$ respectively. The details of characterizing target/clutter scattering by their covariance matrix can be found in [17].

Our goal is to estimate $\mathbf{x}_t$ from $\mathbf{y}$. The clutter and noise term $\mathbf{A}\mathbf{x}_{\rm c}+\mathbf{v}$ has zero-mean and variance

$$\begin{aligned} &E[(\mathbf{A}\mathbf{x}_{\rm c} + \mathbf{v})(\mathbf{A}\mathbf{x}_{\rm c} + \mathbf{v})] \\ &= \mathbf{A}\left[E\left(\mathbf{x}_{\rm c}\mathbf{x}_{\rm c}^T\right)\right]\mathbf{A}^H + E(\mathbf{v}\mathbf{v}^T) \\ &= \mathbf{A}\mathbf{C}_c\mathbf{A}^H + \sigma_v^2\mathbf{I}_m. \end{aligned}$$

Notice that (7) is a linear Gaussian model. In addition, $\mathbf{x}_t$ and $\mathbf{A}\mathbf{x}_{\rm c}+\mathbf{v}$ are uncorrelated. Thus, using [18, Theorem 12.1], we obtain that the minimum MSE $\mathbf{D}$ of estimating $\mathbf{x}_t$ from $\mathbf{y}$ satisfies

$$\mathbf{D}^{-1} = \mathbf{C}_t^{-1} + \mathbf{A}^H(\mathbf{A}\mathbf{C}_c\mathbf{A}^H + \sigma_v^2\mathbf{I}_m)^{-1}\mathbf{A}. \tag{8}$$

We also impose P as the average power constraint on the m transmitted signals. This leads to the following condition

$$\sum_{i=1}^m \|\mathbf{a}_i\|^2 \le mP.$$

In the form of $\mathbf{A}$, the power constraint can be rewritten as

$$\sum_{i=1}^m \|\mathbf{a}_i\|^2 = \mathrm{Tr}(\mathbf{A}\mathbf{A}^H) \le mP. \tag{9}$$

Therefore, to choose the optimal polarization and power scheduling to minimize the MSE of estimating $\mathbf{S}_{\rm t}$ subject to the average power constraint P using m diversely polarized pulses, we obtain the following optimization problem

$$\begin{aligned} \min_{\mathbf{A},\mathbf{D}} \quad & \mathrm{Tr}(\mathbf{D}) \\ \text{s.t.} \quad & \mathbf{D}^{-1} = \mathbf{C}_t^{-1} + \mathbf{A}^H\left(\mathbf{A}\mathbf{C}_c\mathbf{A}^H + \sigma_v^2\mathbf{I}_m\right)^{-1}\mathbf{A} \\ & \mathrm{Tr}(\mathbf{A}\mathbf{A}^H) \le mP \\ & \mathbf{D} \succeq 0 \end{aligned} \tag{10}$$

The above problem is apparently neither linear nor convex in $\mathbf{A}$ or $\mathbf{D}$. In the next section, we reformulate the above problem in a convex form to make it efficiently solvable.

III. Convex Reformulation

In this section we recast (10) in a convex form and then solve it numerically using existing solvers for convex optimization problems. Let us first introduce the following new variables

$$\begin{aligned} \mathbf{B} &= \sigma_v^{-1}\mathbf{A}\mathbf{C}_c^{1/2} \\ \mathbf{C}_{t,1}^{-1} &= \mathbf{C}_c^{1/2}\mathbf{C}_t^{-1}\mathbf{C}_c^{1/2} \\ \mathbf{D}_1^{-1} &= \mathbf{C}_c^{1/2}\mathbf{D}^{-1}\mathbf{C}_c^{1/2} \end{aligned}$$

In terms of the new variables, (8) becomes

$$\mathbf{D}_1^{-1} = \mathbf{C}_{t,1}^{-1} + \mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \mathbf{I}_m)^{-1}\mathbf{B}. \tag{11}$$

Using the relation $\mathbf{B} = \sigma_v^{-1}\mathbf{A}\mathbf{C}_c^{1/2}$, we can also calculate

$$\begin{aligned} \mathrm{Tr}(\mathbf{A}\mathbf{A}^H) &= \mathrm{Tr}\left(\sigma_v\mathbf{B}\mathbf{C}_c^{-1/2}\mathbf{C}_c^{-H/2}\mathbf{B}^H\sigma_v\right) \\ &= \sigma_v^2\,\mathrm{Tr}\left(\mathbf{B}\mathbf{C}_c^{-1}\mathbf{B}^H\right) \\ &= \sigma_v^2\,\mathrm{Tr}\left(\mathbf{C}_c^{-1}\mathbf{B}^H\mathbf{B}\right). \end{aligned}$$

This leads to the following power constraint on $\mathbf{B}$ (cf. (9)):

$$\mathrm{Tr}\left(\mathbf{C}_c^{-1}\mathbf{B}^H\mathbf{B}\right) \le mP/\sigma_v^2.$$

To further simplify (11), we apply the matrix inversion lemma and obtain

$$(\mathbf{I}_4 + \mathbf{B}^H\mathbf{B})^{-1} = \mathbf{I}_4 - \mathbf{B}^H(\mathbf{I}_m + \mathbf{B}\mathbf{B}^H)^{-1}\mathbf{B}.$$

Therefore,

$$\begin{aligned} \mathbf{D}_1^{-1} &= \mathbf{C}_{t,1}^{-1} + \mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \mathbf{I}_m)^{-1}\mathbf{B} \\ &= \mathbf{C}_{t,1}^{-1} + \mathbf{I}_4 - (\mathbf{I}_4 + \mathbf{B}^H\mathbf{B})^{-1}. \end{aligned} \tag{12}$$

Notice that $\mathbf{D}_1^{-1} = \mathbf{C}_c^{1/2}\mathbf{D}^{-1}\mathbf{C}_c^{1/2}$, which implies $\mathrm{Tr}(\mathbf{D}) = \mathrm{Tr}(\mathbf{C}_c^{1/2}\mathbf{D}_1\mathbf{C}_c^{1/2}) = \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c)$. Therefore, in terms of $\mathbf{D}_1$ and $\mathbf{B}$, we can recast the optimization problem (10) as

$$\begin{aligned} \min_{\mathbf{B},\mathbf{D}_1} \quad & \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c) \\ \text{s.t.} \quad & \mathbf{D}_1^{-1} = \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - (\mathbf{I}_4 + \mathbf{B}^H\mathbf{B})^{-1} \\ & \mathrm{Tr}\left(\mathbf{C}_c^{-1}\mathbf{B}^H\mathbf{B}\right) \le mP/\sigma_v^2 \\ & \mathbf{D}_1 \succeq 0 \end{aligned} \tag{13}$$

It is then natural to introduce a positive semi-definite matrix $\mathbf{R} \stackrel{\text{def}}{=} \mathbf{B}^H\mathbf{B}$. We further change the first constraint (13) into inequality.[1] Applying these change on (13) we obtain

$$\begin{aligned} \min_{\mathbf{R},\mathbf{D}_1} \quad & \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c) \\ \text{s.t.} \quad & \mathbf{D}_1^{-1} \preceq \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - (\mathbf{I}_4 + \mathbf{R})^{-1} \\ & \mathrm{Tr}(\mathbf{C}_c^{-1}\mathbf{R}) \le mP/\sigma_v^2 \\ & \mathbf{R} \succeq 0, \quad \mathbf{D}_1 \succeq 0. \end{aligned}$$

[1]Notice that this does not change the solution since at the optimal solution, the equality holds, which can be proved by complementary slackness theorem [19].

Introducing another auxiliary semidefinite matrix $\mathbf{S}$, we can write the first constraint equivalently as two inequalities:

$$\mathbf{D}_1^{-1} \preceq \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - \mathbf{S}$$
$$\mathbf{S} \succeq (\mathbf{I}_4 + \mathbf{B}^H\mathbf{B})^{-1}$$

By Schur's complement, the above two inequalities can be changed into the following convex form:

$$\begin{bmatrix} \mathbf{D}_1 & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - \mathbf{S} \end{bmatrix} \succeq 0$$
$$\begin{bmatrix} \mathbf{S} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{R} \end{bmatrix} \succeq 0$$

Eventually we obtain the following convex programming

$$\begin{aligned} \min_{\mathbf{R},\mathbf{S},\mathbf{D}_1} \quad & \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{D}_1 & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - \mathbf{S} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} \mathbf{S} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{R} \end{bmatrix} \succeq 0 \\ & \mathrm{Tr}\left(\mathbf{C}_c^{-1}\mathbf{R}\right) \leq mP/\sigma_v^2 \\ & \mathbf{R} \succeq 0 \end{aligned} \tag{14}$$

We introduce $\mathbf{Q} \overset{\text{def}}{=} \mathbf{A}^H\mathbf{A}$, we obtain that $\mathbf{R} = \mathbf{B}^H\mathbf{B} = \sigma_v^{-2}\mathbf{C}_c^{1/2}\mathbf{A}^H\mathbf{A}\mathbf{C}_c^{1/2} = \sigma_v^{-2}\mathbf{C}_c^{1/2}\mathbf{Q}\mathbf{C}_c^{1/2}$. We can thus write (14) in terms of $\mathbf{Q}$ as below:

$$\begin{aligned} \min_{\mathbf{R},\mathbf{S},\mathbf{D}_1} \quad & \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{D}_1 & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - \mathbf{S} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} \mathbf{S} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \sigma_v^{-2}\mathbf{C}_c^{1/2}\mathbf{Q}\mathbf{C}_c^{1/2} \end{bmatrix} \succeq 0 \\ & \mathrm{Tr}(\mathbf{Q}) \leq mP \\ & \mathbf{Q} \succeq 0 \end{aligned} \tag{15}$$

Problem (15) is a semidefinite programming (SDP) [19]. SDP is a special class of convex optimization problem, and therefore enjoys all the advantages of convexity. There are well-developed numerical methods to solve a general convex optimization problem, among which the most well known one is the interior point method. In the numerical example, we adopt an optimization toolbox: SeDuMi[2] [20] to solve the SDP formulated in (15).

After solving the optimal $\mathbf{Q}$, we can decompose it to get $\mathbf{A}$ using the fact that $\mathbf{Q} \overset{\text{def}}{=} \mathbf{A}^H\mathbf{A}$. Assuming the positive semidefinite matrix $\mathbf{Q}$ has the following eigendecomposition:

$$\mathbf{Q} = \mathbf{U}_{\mathbf{Q}}^H\mathbf{\Lambda}_{\mathbf{Q}}\mathbf{U}_{\mathbf{Q}},$$

where $\mathbf{U}_{\mathbf{Q}}$ is a unitary matrix and $\mathbf{\Lambda}_{\mathbf{Q}}$ is diagonal. If $m = 4$, we have $\mathbf{A} = \mathbf{V}_{4,4}\mathbf{\Lambda}_{\mathbf{Q}}^{1/2}\mathbf{U}_{\mathbf{Q}}$, where $\mathbf{V}_{4,4}$ is any 4×4 unitary matrix. If $m > 4$, we obtain that $\mathbf{A} = \mathbf{V}_{m,4}\mathbf{\Lambda}_{\mathbf{Q}}^{1/2}\mathbf{U}_{\mathbf{Q}}$, where $\mathbf{V}_{m,4}$ is any $m \times 4$ matrix whose columns are 4 ortho-normal vectors of dimension m. Based on $\mathbf{A}$, we can obtain the optimal power levels and polarizations for the m pulses using (4) and (6). It is easy to see that the solution of $\mathbf{A}$ is not unique. In addition, when $m > 4$, one of the optimal solutions is to transmit the first four pulses only using the total power budget.[3] This corresponds to the choice of $\mathbf{V}_{m,4} = [\mathbf{I}_4, \mathbf{0}_{4,m-4}]^T$. Thus, without losing the optimality, we assume $m = 4$ in the numerical simulations in the subsequent sections.

[2]SeDuMi, which stands for Self-Dual-Minimisation, is a software package that solves optimization problems over symmetric cones using the primal-dual interior-point methods.

[3]This is under the assumption that maximum transmit power is not of a concern. Otherwise, a constraint can be added into (15) to model the maximum transmit power constraint(s).

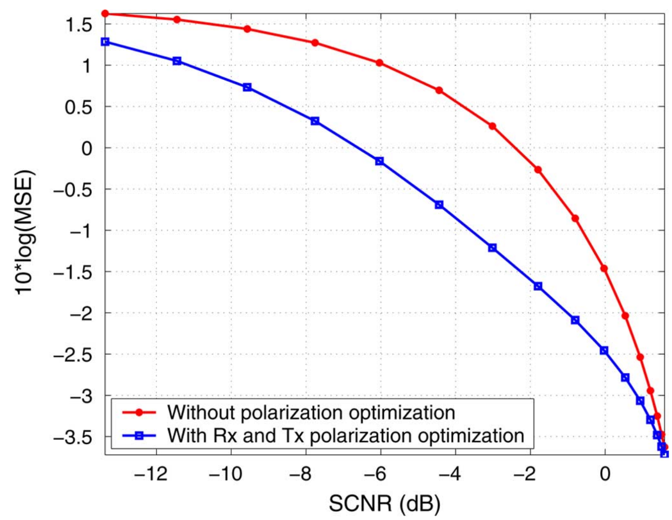

Fig. 1. MSE performance comparison: with or without polarization optimization.

IV. Performance Comparison With Conventional Approach

In this section, We show a numerical example to demonstrate the advantage of the polarization optimization over conventional approach with fixed H and V polarizations. In the conventional system, the transmitter polarization and receiver polarization are either H or V, i.e., $\boldsymbol{\xi} = [1,0]^T$ or $[0,1]^T$, and $\boldsymbol{\eta} = [1,0]^T$ or $[0,1]^T$. With the assumption of equal power for each pulse, this results in the following options for $\mathbf{a}(i)$ (rows of $\mathbf{A}$):

$$\begin{aligned} \mathbf{a}_1^{(c)} &= \sqrt{P}\,\mathbf{p}([1,0]^T,[1,0]^T) = \sqrt{P}\,[1,0,0,0]^T \\ \mathbf{a}_2^{(c)} &= \sqrt{P}\,\mathbf{p}([0,1]^T,[1,0]^T) = \sqrt{P}\,[0,1,0,0]^T \\ \mathbf{a}_3^{(c)} &= \sqrt{P}\,\mathbf{p}([1,0]^T,[0,1]^T) = \sqrt{P}\,[0,0,1,0]^T \\ \mathbf{a}_4^{(c)} &= \sqrt{P}\,\mathbf{p}([0,1]^T,[0,1]^T) = \sqrt{P}\,[0,0,0,1]^T \end{aligned} \tag{16}$$

which results in an $\mathbf{A} = \sqrt{P}\,\mathbf{I}_4$.

Further, we choose the target covariance matrix in the following form:

$$\mathbf{C}_t = \mathbf{U}_t\mathbf{\Lambda}_t\mathbf{\Lambda}_t^H\mathbf{U}_t^H. \tag{17}$$

In the above formula, a unitary matrix is randomly picked for $\mathbf{U}_t$, which is constructed with the left singular vectors of a 4×4 matrix $\mathbf{M}$ with i.i.d. complex Gaussian entries, i.e., the singular value decomposition of $\mathbf{M}$ can be written as $\mathbf{M} = \mathbf{U}_t\mathbf{\Lambda}_{\mathbf{M}}\mathbf{U}_r$. We also choose $\mathbf{\Lambda}_t = \mathrm{diag}([0.1, 0.1, 0.3, 1])$. Similarly, the clutter covariance matrix is chosen as:

$$\mathbf{C}_c = \mathbf{U}_c\mathbf{\Lambda}_c\mathbf{\Lambda}_c^H\mathbf{U}_c^H, \tag{18}$$

in which $\mathbf{\Lambda}_c = \mathrm{diag}([0.25, 0.25, 0.25, 0.25])$, and $\mathbf{U}_c$ is chosen similarly as $\mathbf{U}_t$, i.e., the left singular vectors of a 4×4 matrix with i.i.d. complex Gaussian entries.

Fig. 1 plots the MSE performance of estimating $\mathbf{S}_t$ based on two schemes: (i) optimally designed $\mathbf{a}(i)$, and (ii) conventional $\mathbf{a}(i)$ from (16). The abbreviation SCNR denotes the signal to clutter and noise ratio. The MSE is calculated by averaging over 100 independent realizations of $\mathbf{G}_t$ and $\mathbf{G}_c$ in (17) and (18) respectively. As can be seen, the optimally designed $\mathbf{a}(i)$ based on polarization selection and power scheduling leads to a power gain of 4–6 dBs.

V. Comparison With Only Transmitter or Receiver Polarization Optimization

In this section, we compare the scattering estimation performance of the joint transmitter/receiver polarization optimization with the cases when there is optimization at either the transmitter or the receiver only.

A. Transmitter Polarization Optimization

For the case of transmitter polarization optimization only, we allow $[\xi_{\rm h}, \xi_{\rm v}]^T$ to be chosen freely but the receive polarizations are fixed to be either [1,0] or [0,1]. As such, recalling (4), we obtain that the rows of $\mathbf{A}$ have the following form:

$$\begin{aligned} \mathbf{a}_1(P_1; \boldsymbol{\xi}_1, [1,0]^T) &= \sqrt{P_1}\left[\xi_{\rm h}^{(1)} \quad 0 \quad 0 \quad \xi_{\rm v}^{(1)}\right]^T \\ \mathbf{a}_2(P_2; \boldsymbol{\xi}_2, [1,0]^T) &= \sqrt{P_2}\left[\xi_{\rm h}^{(2)} \quad 0 \quad 0 \quad \xi_{\rm v}^{(2)}\right]^T \\ \mathbf{a}_3(P_3; \boldsymbol{\xi}_3, [0,1]^T) &= \sqrt{P_3}\left[0 \quad \xi_{\rm v}^{(3)} \quad \xi_{\rm h}^{(3)} \quad 0\right]^T \\ \mathbf{a}_4(P_4; \boldsymbol{\xi}_4, [0,1]^T) &= \sqrt{P_4}\left[0 \quad \xi_{\rm v}^{(4)} \quad \xi_{\rm h}^{(4)} \quad 0\right]^T. \end{aligned} \tag{19}$$

We shall analyze how to incorporate the constraints on $\mathbf{A}$ in (19) into the optimization problem in (15). Recalling that $\mathbf{Q} = \mathbf{A}^H\mathbf{A}$. It is easy to see that the resulted $\mathbf{A}$ from (19) leads to an $\mathbf{Q}$ with block diagonal structure:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}^T \tag{20}$$

where both $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are 2×2 positive semidefinite matrices. In addition, for every positive semidefinite block diagonal matrix $\mathbf{Q}$ as given in (20), there is a unique $\mathbf{A}$ with the row structure in (19) satisfying $\mathbf{A}^H\mathbf{A} = \mathbf{Q}$.

Therefore, we can add constraint (20) into (15) to solve for the optimal transmitter polarization optimization. As a summary, we obtain the following problem:

$$\begin{aligned} \min_{\mathbf{R},\mathbf{S},\mathbf{D}_1} \quad & \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{D}_1 & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - \mathbf{S} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} \mathbf{S} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \sigma_v^{-2}\mathbf{C}_c^{1/2}\mathbf{Q}\mathbf{C}_c^{1/2} \end{bmatrix} \succeq 0 \\ & \mathrm{Tr}(\mathbf{Q}) \leq mP \\ & \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}^T \succeq 0. \end{aligned} \tag{21}$$

Compared to (15), there is an additional block diagonal constraint on $\mathbf{Q}$. After solving the optimal $\mathbf{Q}$, we can decompose it to get an $\mathbf{A}$ with row structure given in (19), from which we can obtain the optimal transmit polarizations $\{\boldsymbol{\xi}_i : 1 \leq i \leq 4\}$ and the power levels $\{P_i : 1 \leq i \leq 4\}$ as well.

B. Receiver Polarization Optimization

Similarly, for the case of receiver polarization optimization, we optimize $[\eta_{\rm h}, \eta_{\rm v}]^T$ but keep the transmitter polarizations to be either [1,0] or [0,1]. In addition, P_i is chosen to be P for all transmitted waveforms. Recalling (4), we obtain that

$$\begin{aligned} \mathbf{a}_1(P; [1,0]^T, \boldsymbol{\eta}_1) &= \sqrt{P}\left[\eta_{\rm h}^{(1)} \quad 0 \quad \eta_{\rm v}^{(1)} \quad 0\right]^T \\ \mathbf{a}_2(P; [1,0]^T, \boldsymbol{\eta}_2) &= \sqrt{P}\left[\eta_{\rm h}^{(2)} \quad 0 \quad \eta_{\rm v}^{(2)} \quad 0\right]^T \\ \mathbf{a}_3(P; [0,1]^T, \boldsymbol{\eta}_3) &= \sqrt{P}\left[0 \quad \eta_{\rm v}^{(3)} \quad 0 \quad \eta_{\rm h}^{(3)}\right]^T \\ \mathbf{a}_4(P; [0,1]^T, \boldsymbol{\eta}_4) &= \sqrt{P}\left[0 \quad \eta_{\rm v}^{(4)} \quad 0 \quad \eta_{\rm h}^{(4)}\right]^T. \end{aligned} \tag{22}$$

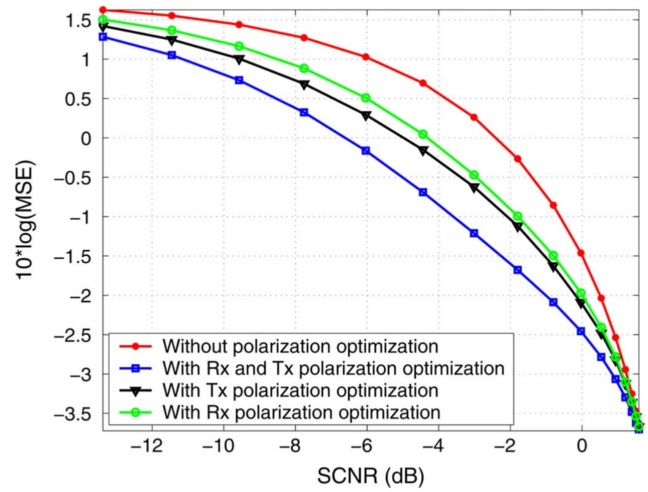

Fig. 2. MSE performance comparison among 4 levels of polarization optimization.

Similarly, the resulted $\mathbf{A}$ from above leads to an $\mathbf{Q}$ with block diagonal structure as in (20), and in addition, it holds that $\mathrm{diag}(\mathbf{Q}) = P\mathbf{I}_4$. It is also easy to see that for every positive semidefinite block diagonal matrix $\mathbf{Q}$ as given in (20) with $\mathrm{diag}(\mathbf{Q}) = P\mathbf{I}_4$, there is a unique $\mathbf{A}$ with the row structure in (19) satisfying $\mathbf{A}^H\mathbf{A} = \mathbf{Q}$.

Therefore, we can add the clock diagonal constraints in (20) and the constraint $\mathrm{diag}(\mathbf{Q}) = P\mathbf{I}_4$ to (15) and obtain the following optimization problem:

$$\begin{aligned} \min_{\mathbf{R},\mathbf{S},\mathbf{D}_1} \quad & \mathrm{Tr}(\mathbf{D}_1\mathbf{C}_c) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{D}_1 & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{C}_{t,1}^{-1} - \mathbf{S} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} \mathbf{S} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{I}_4 + \sigma_v^{-2}\mathbf{C}_c^{1/2}\mathbf{Q}\mathbf{C}_c^{1/2} \end{bmatrix} \succeq 0 \\ & \mathrm{Tr}(\mathbf{Q}) \leq mP \\ & \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}^T \succeq 0 \\ & \mathrm{diag}(\mathbf{Q}) = P\mathbf{I}_4. \end{aligned} \tag{23}$$

By solving for the optimal $\mathbf{Q}$ from the above problem, we can obtain the optimal receiver polarizations $\{\boldsymbol{\eta}_i : 1 \leq i \leq 4\}$ by suing the relationship between $\mathbf{Q} = \mathbf{A}^H\mathbf{A}$ and (22).

Fig. 2 plots the MSE performance of estimating $\mathbf{S}_t$ based on four schemes: (i) jointly optimally designed transmitter/receiver polarization, (ii) transmitter polarization optimization only, (iii) receiver polarization optimization only, and (iv) conventional approach with fixed transmitter and receiver polarizations as in (16). Similar to Fig. 1, the MSE is calculated by averaging over 100 independent realizations of $\mathbf{G}_t$ and $\mathbf{G}_c$ (cf., (17) and (18). We can see that optimal designed polarizations at either the transmitter or the receiver improves the estimation performance. In addition, compared to the receiver polarization optimization, the optimization at the transmitter leads to slightly larger performance improvement. This is due to the fact that at the transmitter, we not only optimize the pulse polarizations but also the pulse power levels, while for the optimization at the receiver side, the power levels among pulses are scheduled equally by the transmitter.

An additional simulation with varying clutter strength is given in Fig. 3. In this figure, we choose $\mathbf{C}_t$ the same as in (17). However, the

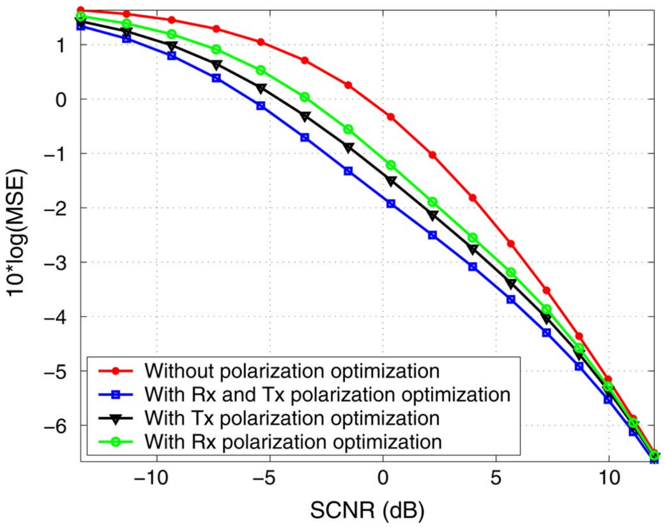

Fig. 3. MSE performance comparison in the case of varying clutter strength.

clutter covariance matrix is chosen to be $\alpha \mathbf{C}_c$ where $\mathbf{C}_c$ is given in (18). The noise power $\sigma_v^2 = 1$. We vary α to obtain different SCNR values.

We shall point out that due to the symmetric property, the performance improvement brought by the polarization optimization at the receiver is the same as that brought by the polarization optimization at the transmitter side. Thus, from Fig. 2 we can see that power allocation (on top of polarization optimization) at the transmitter leads to additional performance gain. Such gain largely depends on the eigen-value distributions of $\mathbf{C}_t$ and $\mathbf{C}_c$.

VI. Conclusion

We investigated the polarization optimization and power scheduling for the estimation of a target scattering matrix in a clutter environment. An average power constraint for radar pulses was assumed. We cast this problem as a nonlinear optimization problem for the optimal design of a radar sensing matrix, which we further reformulated into a convex form and is thus numerically easily solvable. The numerical results demonstrate that by carefully choosing the transmitter/receiver polarizations and pulse power levels, clutter interference can be efficiently suppressed.

We proposed a one-step optimization to select the optimal polarizations and power levels for all pulses. It is expected that additional performance gain could be achieved if such optimization is done sequentially on a pulse-by-pulse basis by using the most currently acquired information. In addition, to make the radar polarimetric sensing more efficient, multi-dimensional information of the incoming EM field at the radar receiver can be simultaneously measured using, e.g., EM vector sensors [11]. Such a strategy will lead to additional constraints on the design of the sensing matrix $\mathbf{A}$. We will explore these topics in our future research.

Throughout this work, we have assumed that the clutter has a Gaussian distribution. For the case of non-Gaussian clutter model, our approach applies if we limit the optimization within the class of linear estimators. We will study in our future work how to extend our joint transmitter and receiver polarization optimization to more general estimator classes.

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