

# MIMO-OFDM Channel Estimation Using Golay Complementary Sequences

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**Abstract**—We present a pilot-assisted method for estimating the frequency selective channel in a MIMO-OFDM (Multiple Input Multiple Output - Orthogonal Frequency Division Multiplexing) system. The pilot sequence is designed using the DFT (Discrete Fourier Transform) of the Golay complementary sequences. Novel exploitation of the perfect autocorrelation property of Golay complementary sequences, in conjunction with OSTBC (Orthogonal Space Time Block Code) based pilot waveform scheduling across multiple OFDM frames, facilitates simple separation of the channel mixtures at the receive antennas. The DFT length used to transform the complementary sequence into the frequency domain is shown to be a key critical parameter for correctly estimating the channel. This channel estimation scheme is then extended to antenna arrays of arbitrary sizes.

## I. INTRODUCTION

OFDM (Orthogonal Frequency Division Multiplexing) is a multi-carrier modulation scheme that has gained considerable popularity over the past decade because of its ability to combat frequency-selective fading normally encountered in a multi-path wireless environment. OFDM converts the frequency selective channel into flat-fading sub-channels, thereby significantly reducing the receiver complexity by eliminating the need for using equalization at the receiver. [1]. MIMO (Multiple Input Multiple Output) systems were first introduced in [2],[3]. Under certain conditions [3], the capacity of MIMO systems is shown to increase linearly with  $\min\{N_T, N_R\}$  where  $N_T$  and  $N_R$  are the number of antennas at the transmitter and the receiver, respectively.

MIMO-OFDM systems provide performance gains because they combine the diversity and multiplexing gains of MIMO with the resilience of OFDM against multi-path fading. In order to achieve these performance improvements, accurate CSI (Channel State Information) is required at the receiver which is obtained via channel estimation. A number of different pilot assisted methods have been proposed in the literature for estimating the channel, such as [4]-[6]. In a pilot assisted scheme for channel estimation, known pilot sub-carriers are inserted in the transmitted signal to sample the multi-path channel. At the receiver, these samples are then used to obtain a representation of the entire channel through linear or higher order interpolation. Pilots essentially sample the frequency

response of the channel. If the sampling is done according to the Nyquist criterion [7], the entire frequency response of the channel can be reconstructed from the samples obtained from the pilots.

The difficulty in channel estimation for multi-antenna systems lies in the fact that at every receive antenna, we get a signal coming from different transmit antennas, and hence different channels. Therefore, we need to estimate the individual channels given mixtures of the channels. A technique for estimating the channel in a  $2 \times 1$  transmit diversity system has been proposed in [8], which is an iterative technique. It requires the initialization of the channel estimates at the receiver by sending the Golay complementary sequences [9] – no data – during the first two OFDM symbol periods followed by a successive interference cancellation procedure that kind of goes back and forth between data estimation and channel estimation. This is because the method in [8] is not based on pilot tones as the Golay codes are simply added to the OFDM frame. In this paper, we present a simpler and computationally efficient method to obtain the sampled channel estimates using pilots without any need for initialization symbols or extra processing at the receiver. Our method uses Golay complementary sequences to design pilots in the frequency domain. Complementary sequences have the property that the sum of their autocorrelation produces a Dirac-delta function. The use of a combination of complementary sequences and OSTBCs (Orthogonal Space Time Block Codes) [10] like coding allows us to separate channels from different transmitting antennas at the receiving antenna.

The paper is organized as follows: section II introduces the channel model, complementary sequences and relevant results about waveform separation in the radar literature. Section III describes the proposed channel estimation paradigm. Simulation results are presented in section IV and section V provides a short discussion and conclusions.

## II. PRELIMINARIES

In this section, we introduce the multi-path channel model, Golay complementary sequences and some relevant literature from radar waveform design.

### A. Multipath Channel Model

We consider a multi-path channel with  $B_c < B_T$ , Where  $B_T$  is the transmission bandwidth and  $B_c$  is the coherence bandwidth.  $B_c$  can be expressed in terms of the maximum delay spread  $\tau_{\max}$  as

$$B_c \approx \frac{1}{\tau_{\max}} \quad (1)$$

We will assume that the channel is slowly fading and is quasi-static over  $n \geq 2$  consecutive symbols. With these assumptions, the sampled channel from transmit antenna  $i$  to receive antenna  $j$  can be represented in the time-domain as

$$h_{ij}(t) = \sum_{l=0}^{L-1} \beta_{ij}^{(l)} \delta(t - t_l) \quad (2)$$

where  $\beta_{ij}^{(l)}$  are i.i.d complex Gaussian random variables with variance  $\sigma_h^2$ .  $L$  is the maximum length of the sampled channel impulse response (CIR) which captures all the necessary variation in the continuous channel impulse response.

### B. Golay Complementary Sequences

A pair of sequences  $e_1[n]$  and  $e_2[n]$  of length  $N_c$  satisfy the Golay property if the sum of their autocorrelation functions satisfy

$$R_{e_1 e_1}[l] + R_{e_2 e_2}[l] = \begin{cases} 2N & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases} \quad (3)$$

for  $l = -N_c - 1, \dots, N_c - 1$ . If we take the DFT of the above equation, we get

$$|E_1[k]|^2 + |E_2[k]|^2 = 2N \quad (4)$$

In order for the above relation to reflect the DFT of (4), The DFT length needs to be chosen properly. This is because  $R_{e_i e_i}[n]$  represents a linear convolution of  $e_i[n]$  and  $e_i^*[-n]$ , i.e.

$$R_{e_1 e_1}[n] = e_1[n] * e_1^*[-n] \quad (5)$$

However, we observe that

$$IDFT_{N_{DFT}} \left\{ |E_i[k]|^2 \right\} = e_i[n] \circledast e_i^*[-n] \quad (6)$$

gives the circular convolution of  $e_i[n]$  and  $e_i^*[-n]$ . Therefore, in order to get (5) from (6), the DFT length  $N_{DFT}$  needs to be chosen such that

$$N_{DFT} \geq 2N_c - 1 \quad (7)$$

This is equivalent to zero padding the original sequence  $e_i[n]$  by  $N_c - 1$  zeros so that their circular convolution represents the linear convolution of the original sequence  $e_i[n]$ . This property will play a key role in our channel estimation strategy, as we later show. Also, we observe that if  $e_1[n]$  and  $e_2[n]$  satisfy the Golay property, then  $e_1^*[-n]$ ,  $e_2^*[-n]$ ;  $e_1^*[-n]$ ,  $e_2[n]$  and  $e_1[n]$ ,  $e_2^*[-n]$  also form complementary sequence pairs.

### C. Complementary Sequences and Radar Waveform Design

In [11] and [12], instantaneous radar polarimetric techniques were proposed for  $2 \times 2$  and  $4 \times 4$  systems using complemen-

tary sequences. The idea is to schedule the complementary sequence according to the OSTBC scheme in order to estimate the radar returns for improved detection performance.

In [11], the target scattering matrix, given by

$$\mathbf{H} = \begin{bmatrix} h_{11}\delta[n-D] & h_{12}\delta[n-D] \\ h_{21}\delta[n-D] & h_{22}\delta[n-D] \end{bmatrix} \quad (8)$$

where  $D$  is the target delay, was estimated by using the OSTBC complementary sequences of length  $N$  given by

$$\mathbf{E} = \begin{bmatrix} e_1[n] & -e_2^*[-n] \\ e_2[n] & e_1^*[-n] \end{bmatrix}$$

where  $h_{ij}$  represents the radar return from the transmitting antenna  $i$  to receiving antenna  $j$  and the columns and rows of  $\mathbf{E}$  represent the transmitting antennas and symbol periods respectively. For simplicity of exposition, we will assume that  $D = 0$ . The received waveform can be represented as

$$\mathbf{R} = \mathbf{H} * \mathbf{E} + \mathbf{N} \quad (9)$$

where  $\mathbf{N}$  is the white noise matrix. The target scattering matrix can be estimated by observing that

$$\mathbf{E} * \mathbf{E}^H = \alpha \mathbf{I} \delta[n] \quad (10)$$

where  $\alpha$  is a constant given by

$$\alpha = |E_1[k]|^2 + |E_2[k]|^2 = 2N \quad (11)$$

and  $*$  represents the matrix convolution. The matrix  $\mathbf{E}^H$  can be written as

$$\mathbf{E}^H = \begin{bmatrix} e_1^*[-n] & e_2^*[-n] \\ -e_2[n] & e_1[n] \end{bmatrix}$$

If we process the received waveform matrix with  $\mathbf{E}^H$ , we see that

$$\mathbf{R} * \mathbf{E}^H = \alpha \mathbf{H} + \mathbf{N}' \quad (12)$$

where

$$\mathbf{N}' = \mathbf{N} * \mathbf{E}^H \quad (13)$$

is still white because the matrix  $\mathbf{H}$  is orthogonal [13]. From this, we can see that the channel can be estimated without the receiver having any prior information about the channel. This property will be central to the idea of channel estimation presented in the next section.

## III. PILOT DESIGN AND CHANNEL ESTIMATION

We consider a  $2 \times 2$  MIMO-OFDM system with  $N$  sub-carriers of which  $N_p$  are pilot sub-carriers. Each pair of transmit-receive antennas encounters a length  $L$  multi-path channel. We will assume that the channel is slowly fading and is quasi-static over  $n \geq 2$  consecutive symbols. The system cyclically extends every OFDM symbols before transmission by appending the trailing  $N_g$  samples to the beginning of the symbol. We assume  $N_g \geq L$  so that all the sub-carriers remain mutually orthogonal at the receiver [1]. The  $N_p$  point pilot sequence is designed in the frequency domain. We present a development without considering the effects of receiver noise

and will later characterize the noisy estimation performance using the NMSE (Normalized Mean Squared Error) criterion.

### A. Frequency Domain Pilot Design

The pilot sequence is designed in the frequency domain using the DFT of complementary sequences. The  $N_p$  point DFT of the complementary sequence is given by

$$\tilde{S}_i[k] = DFT_{N_p}\{s_i[n]\} \quad (14)$$

where  $N_p$  is the number of pilots. We use  $\tilde{S}[k]$  to emphasize the fact that this represents the pilot sequence with  $k = 0, 1, \dots, N_p$  and is the sequence carried by the pilot sub-carriers. The advantage of designing the pilot sequence in the frequency domain lies in the fact that the sequence is carried by orthogonal sub-carriers. The relative magnitudes of these sub-carriers change because each sub-carrier, upon passing through the channel, is multiplied by the corresponding value of the channel frequency response at that frequency. However, with the assumption that the guard interval  $N_g$  is longer than the maximum length of the sampled channel impulse response  $L$ , the sub-carriers still remain orthogonal at the receiver, and we can recover the pilot sequence without any interference from the data. If  $H_{ij}[k]$  represents the  $N$ -point DFT of the sampled channel impulse response between  $i^{th}$  transmit antenna and the  $j^{th}$  receive antenna, we define the sampled frequency response at the pilot frequencies as

$$\tilde{H}_{ij}[k] = H_{ij}[kN/N_p] \quad k = 0, 1, \dots, N_p - 1 \quad (15)$$

The received sequence on the pilot subcarriers at each receive antenna during the two OFDM symbol intervals, after removing the cyclic prefix and taking the DFT, can be written as

$$\tilde{\mathbf{R}}[k] = \begin{bmatrix} \tilde{\mathbf{Y}}_1[k] \\ \tilde{\mathbf{Y}}_2[k] \end{bmatrix} = \tilde{\mathbf{H}}[k]\tilde{\mathbf{S}}[k] \quad (16)$$

where

$$\tilde{\mathbf{Y}}_i[k] = [ \tilde{H}_{1i}[k] \quad \tilde{H}_{2i}[k] ] \tilde{\mathbf{S}}[k] \quad (17)$$

where  $i$  is the receive antenna index and

$$\tilde{\mathbf{S}}[k] = \begin{bmatrix} \tilde{S}_1[k] & -\tilde{S}_2^*[k] \\ \tilde{S}_2[k] & \tilde{S}_1^*[k] \end{bmatrix} \quad (18)$$

is the OSTBC matrix of transformed complementary sequences. In order to separate the different channels at the receiver, we need to process the received waveforms with a matrix  $\tilde{\mathbf{A}}[k]$  such that

$$\tilde{\mathbf{R}}[k]\tilde{\mathbf{A}}[k] = \alpha\tilde{\mathbf{H}}[k] \quad (19)$$

By exploiting (4), we can directly infer that  $\mathbf{A}[k] = \tilde{\mathbf{S}}^H[k]$  gives the desired result. Therefore, we have that

$$\tilde{\mathbf{Y}}_i[k]\tilde{\mathbf{S}}^H[k] = \alpha [ \tilde{H}_{1i}[k] \quad \tilde{H}_{2i}[k] ] \quad (20)$$

where  $\alpha$  is a constant independent of  $k$ . Notice that the received vector at each received antenna is processed independently of other receive antennas in the system. This property would help us in extending this scheme to antenna arrays of arbitrary sizes, as we later show.

After obtaining the frequency domain channel estimates on the pilot subcarrier frequencies, we take the  $N_p$ -point IDFT of this sequence to get the time domain channel, and then we take the  $N$  point DFT of the time domain channel to get the frequency response of the channel on all sub-carrier frequencies. In the next section, we derive a relationship between  $N_p$ ,  $N_c$ , and  $L$  that would allow us to get the time domain channel from the frequency domain sampled channel on all the subcarrier frequencies.

### B. DFT Length

We saw in (20) that the channel estimation involves a product of three terms:  $S_i[k]$ ,  $H_{ij}[k]$ , and  $S_j^*[k]$ . This operation represents a three-fold circular convolution in the time domain, i.e.

$$\tilde{S}_i[k]\tilde{H}_{ij}[k]\tilde{S}_j^*[k] = DFT \{s_i[n] \otimes h_{ij}[n] \otimes s_j^*[-n]\} \quad (21)$$

where  $s_i[n]$  and  $s_j^*[n]$  are length  $N_c$  sequences and  $h_{ij}[n]$  has length  $L$ . We know that if two sequences of length  $M$  and  $Q$  are linearly convolved, the resulting sequence is of length  $M + Q - 1$ . In order to get the same sequence through DFT processing, we need to take the  $M + Q - 1$  point DFT of both the sequences (by zero padding the two sequences to make them of length  $M + Q - 1$ ), multiply the DFTs together, and take the IDFT to get the  $M + Q - 1$  point linear convolution. This idea can be applied to our three-fold convolution by choosing the initial DFT length to be at least  $2N_c + L - 2$ . However, since convolution satisfies both commutativity and associativity, we can first focus our attention on the convolution of the complementary sequences. The sum of the  $2N_c - 1$  point linear convolution of the complementary sequences is a Dirac delta function delayed by  $N_c - 1$ , i.e.  $\delta[n - N_c - 1]$ . The convolution of channel with this function is

$$h_{ij}[n] * \delta[n - N_c - 1] = h_{ij}[n - N_c - 1] \quad (22)$$

From this, we see that the resulting three-fold convolution is the sampled channel impulse response delayed by  $N_c - 1$ . Note that this resulting  $2N_c + L - 2$  point sequence has  $N_c - 1$  zeros before and after the  $L$  point channel sequence. This tells us that if we were to reduce the initial DFT length to  $N_c + L - 1$ , this would have the effect of aliasing the trailing  $N_c - 1$  zeros in the three-fold linear convolution with the leading  $N_c - 1$  zeros. What we get is an  $N_c + L - 1$  point sequence which represents the actual sampled channel response delayed by  $N_c - 1$ . We can use the fact that this  $N_c + L - 1$  sequence still has  $N_c - 1$  zeros to further reduce the initial DFT length by observing that if we reduce the initial DFT length to  $N_c$ , the resulting three fold convolution would be the initial  $N_c - 1$  zeros being aliased with the values of the sampled channel response, which can also be thought of as a circular shift of the channel response. Since we know  $N_c$  and the maximum channel length  $L$ , we know the amount of circular shift present in the estimated channel, and we can circularly shift it back to get the actual channel response. Therefore, to estimate the channel correctly, we can establish a relationship between the

number of pilots  $N_p$ , which also represents the initial DFT length used to transform the complementary sequences into the frequency domain, the length of the complementary sequences  $N_c$ , and maximum length of the sampled channel impulse response  $L$ , given by

$$N_p \geq N_c \geq L \quad (23)$$

Note that pilots represent an overhead in a communication system, and therefore, we need to minimize the number of pilots needed to achieve a desired performance level and the minimum number of pilots required by our scheme is  $L$  provided that we can find complementary sequences with length  $N_c = L$ .

### C. Extension to Arbitrary Antenna Arrays

We now extend these results to antenna arrays of arbitrary size. To this end, consider a MIMO system with  $m$  transmit and  $p$  receive antennas. We have already seen that the processing at every receive antenna is independent of other receive antennas, so we can focus our attention on the transmit side only. We want to design a space-time scheduling scheme using complementary sequences that would allow us to separate different channels from the mixture of channels at every receive antenna. Consider an  $n \times n$  space time code given by

$$\mathbf{S}_n[k] = \begin{bmatrix} \mathbf{S}_{\frac{n}{2}}[k] & -\mathbf{S}_{\frac{n}{2}}^H[k] \\ \mathbf{S}_{\frac{n}{2}}[k] & \mathbf{S}_{\frac{n}{2}}^H[k] \end{bmatrix} \quad (24)$$

where  $\mathbf{S}_2$  is as given by (7) and  $n = 2^j$ ,  $j \in \mathbb{Z}^+$ . From this, we see that

$$\mathbf{S}_n \mathbf{S}_n^H = \mathbf{S}_n \mathbf{S}_n^H = \beta \mathbf{I}_{n \times n} \quad (25)$$

which can be shown by using the induction hypothesis on  $n$ . This, along with the theory of generalized orthogonal designs [10], shows that if we have an  $m$  transmit antenna system, we need a space-time code of dimension  $m \times n$  where  $n = 2^j$ ,  $j \in \mathbb{Z}^+$  and  $2^{j-1} < m \leq 2^j$ . This is due to the fact that since each row of the matrix  $\mathbf{S}_n[k]$  is orthogonal to every other row, we can extract a sub-matrix  $\mathbf{S}'_m$  of dimension  $m \times n$  with orthogonal rows. Since,

$$\mathbf{S}'_m \mathbf{S}'_m^H = \beta' \mathbf{I}_{m \times m} \quad (26)$$

we can separate the individual channels at every receiver through linear processing by the matrix  $\mathbf{S}'_m^H$ . This shows that our channel estimation scheme works for antenna arrays of arbitrary sizes as long as the channel coherence time  $T_c$  is greater than or equal to  $nT_s$  where  $T_s$  is the OFDM symbol interval.

## IV. SIMULATION RESULTS

We consider an OFDM system with  $N = 256$  sub-carriers and  $N_p = 16$  equi-spaced pilot sub-carriers. The channel is a unit variance Rayleigh fading channel with uniform power delay profile and  $L = 5$  taps. We use real valued

complementary sequence of length  $N_c = 10$ , given by

$$\begin{aligned} s_1[n] &= \{1, 1, -1, 1, -1, 1, -1, -1, 1, 1\} \\ s_2[n] &= \{1, 1, -1, 1, 1, 1, 1, 1, -1, -1\} \end{aligned} \quad (27)$$

Our performance measure is the  $NMSE$  between the actual channel impulse response and the estimated channel response, i.e.

$$J_{NMSE}(SNR_{ave}) = \frac{\sum_{l=0}^{L-1} |h_{ij}[l] - \hat{h}_{ij}[l]|^2}{\sum_{l=0}^{L-1} |h_{ij}[l]|^2} \quad (28)$$

$SNR_{ave}$  is the received  $SNR$  averaged over the two antennas and symbol periods given by

$$SNR_{ave} = \frac{1}{4} \sum_{i=1}^2 \sum_{j=1}^2 SNR_{ij} \quad (29)$$

Where  $SNR_{ij}$  is the  $SNR$  at the  $i^{th}$  receive antenna in the  $j^{th}$  symbol interval. We showed in the previous section that we can get the exact channel estimates in the case where there is no noise in the system. Therefore, the performance is limited only by the receiver noise. Figure 1 shows the  $NMSE$  plotted against  $SNR_{ave}$ . The  $NMSE$  decreases monotonically as the receiver  $SNR$  increases, thereby validating our claim that the estimation performance is limited only by the receiver noise. In Figure 2, we show the estimated and the actual

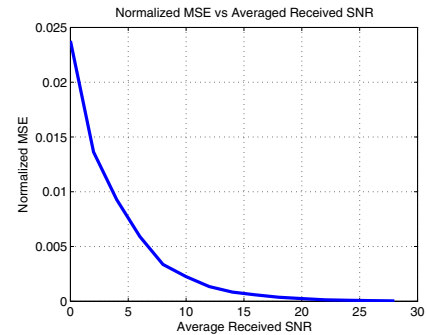


Fig. 1. Plot of NMSE against the average  $SNR$

channel impulse response for the channel from the transmit antenna 1 to receive antenna 1 in the case where there is no noise and we have  $N_p \geq N_c + L$ . The impulse responses are exactly the same, except that the estimated channel response is delayed by  $N_c - 1$  samples. This happens because the autocorrelation on the complementary sequences is a delta function with a delay of  $N_c - 1$ , and since the estimation of the channel impulse response involves the convolution of the actual channel response with the autocorrelation function of the complementary sequences, we observe a delay of  $N_c - 1$  samples. In Figure 3, the estimated and the actual channel impulse responses are shown for the case where  $N_p < N_c + L - 1$ . Specifically, we have  $N_p = 16$ ,  $N_c = 10$ , and  $L = 10$  and there is no noise. We can see from this figure that the estimated channel impulse response is a time-domain aliased version of the actual channel impulse response,

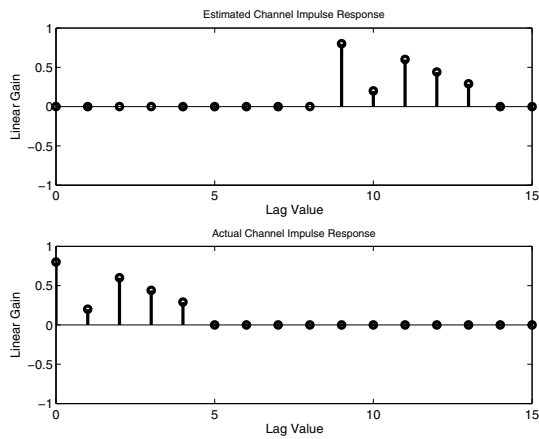


Fig. 2. Estimated and Actual CIR

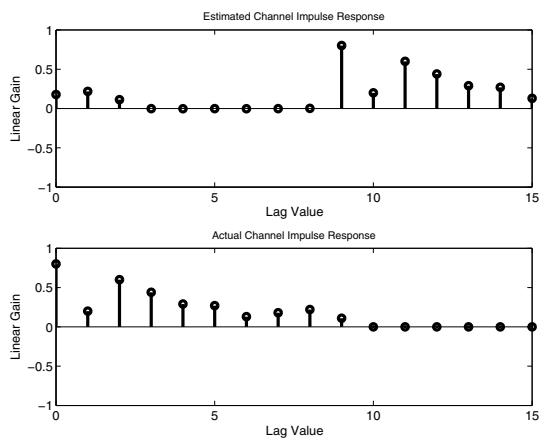


Fig. 3. Estimated and Actual CIR for  $N_p < N_c + L - 1$

where the time-domain aliasing manifests itself as a circular shift of the channel impulse response. However, since we have  $N_p \geq N_c$ , if we left shift the sequence by  $N_c + L - 1 - N_p = 3$ , we have the same situation as the one shown in Figure 2, where the estimated channel is just a delayed version of the actual channel.

## V. CONCLUSIONS

We have introduced a new channel estimation technique that uses transformed Golay complementary sequences based pilot waveforms. The OSTBC based scheduling of these waveforms facilitates simple separation of the channel mixtures in a MIMO environment when certain constraints on the DFT size used to transform the complementary sequences into the frequency domain are met. In systems where the maximum sampled channel length does not exceed the cyclic prefix, it has been shown that the performance of this scheme is limited only by the receiver noise if we assume perfect timing and frequency synchronization. We have provided a method to extend this channel estimation scheme for use in antenna arrays of arbitrary sizes. Simulation results have been provided to confirm the analytical results.

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