Cramér-Rao Lower Bounds for the Joint Estimation of Target Attributes Using MIMO Radar

Jun Jason Zhang†, Ghassan Maalouli†, Antonia Papandreou-Suppappola† and Darryl Morrell‡

†SenSIP Center, Department of Electrical Engineering, Arizona State University, Tempe, AZ
‡Department of Engineering, Arizona State University, Polytechnic Campus, Mesa, AZ

E-mail: Jun.Zhang.EE@asu.edu, g.maalouli@ieee.org, papandreou@asu.edu, morrell@asu.edu

***Abstract*— We derive the Cramér-Rao lower bound (CRLB) on the covariance of the joint estimates of the parameters of moving targets using measurements from multiple-input, multiple-output (MIMO) radars. We first derive the CRLB for MIMO radars with colocated antennas for estimating the target's direction of arrival, range and range-rate. We then demonstrate that the CRLB for phased array radars is a special case of the CRLB for MIMO radars with colocated antennas. We also derive the CRLB for MIMO radars with widely-separated antennas for estimating the target's location and velocity, and we compare it to the CRLB for the multistatic radar. For both types of MIMO radar systems, we show that the CRLB is related to the parameters of the transmitted waveforms, as demonstrated with numerical simulations.**

I. Introduction

A MIMO radar system can transmit multiple and different waveforms, and it has the potential to achieve diversity and increase target estimation performance. With colocated antennas, an increase in estimation and beamforming performance can be obtained using waveform diversity [1]–[5]. When the transmitter and/or receiver antennas are widely-separated, target detection and estimation performance can be increased using space diversity [6]–[9]. Studies on estimating the CRLB of specific target parameters using different assumptions have been considered in [2], [3], [6], [8], [10], [11]. Also, the MIMO radar ambiguity function (AF) and related CRLB derivations have been discussed in [10]–[13]. Note that knowledge of the joint CRLB on the covariance of target parameters, together with the use of multiple waveforms, would lead to promising new results on waveform design and diversity and increase target localization, detection, and tracking performance [10], [11].

In this work, we derive the CRLB to jointly estimate the attributes of a moving target using MIMO radar. Specifically, we develop the CRLB both for colocated and widely-separated antennas. For both cases, we demonstrate the relationship of the CRLB to the transmitted waveform and discuss the possible use of the CRLB to exploit waveform diversity.

This work was supported in part by the Department of Defense Grant No. AFOSR FA9550-05-1-0443.

II. CRLB With Colocated Antenna MIMO Radars

A. Signal and Noise Model

We consider the MIMO radar system depicted in Figure 1 that consists of N_T transmit and N_R receive colocated antennas. The signal $s_i(t)$ transmitted by the ith antenna, $i = 1, \ldots, N_T$, is sampled every T_s to obtain the nth snapshot $s_i[n] = s_i(nT_s)$, $n = 1, \ldots, N$. The signal is time-shifted by τ and frequency-shifted by ν when reflected by a point-source target. We form the $N_T \times N$ signal matrix $\mathbf{S}(\tau) = [\mathbf{s}_1(\tau) \ \cdots \ \mathbf{s}_{N_T}(\tau)]^{\mathbb{T}}$, where $\mathbf{s}_i(\tau) = [s_i[1;\tau] \ \cdots s_i[N;\tau]]^{\mathbb{T}}$, $s_i[n;\tau]$ is the sampled delayed transmitted signal $s_i(t-\tau)$, and $\mathbb{T}$ denotes vector transpose. The received signal is represented by the $N_R \times N$ matrix $\mathbf{R} = \beta\,\mathbf{A}(\theta)\,\mathbf{S}(\tau)\,\mathbf{D}(\nu) + \mathbf{W}$, where $\mathbf{A}(\theta) \stackrel{\triangle}{=} \mathbf{a}(\theta)\,\mathbf{v}^{\mathbb{T}}(\theta)$, θ is the target direction of arrival (DOA), $\mathbf{v}(\theta)$ and $\mathbf{a}(\theta)$ are the length N_T and N_R transmitter and receiver antenna steering vectors, respectively, β is the target reflection coefficient assumed unknown and deterministic during the coherent processing of the N snapshots, $\mathbf{W}$ is the $N_R \times N$ noise matrix whose (k,n)th element is the nth snapshot of additive zero-mean Gaussian noise of the kth receiver antenna, and $\mathbf{D}(\nu)$ is a diagonal matrix whose elements $e^{j2\pi\nu n T_s/N}$, $n = 1, \ldots, N$, correspond to N frequency shifts [1].

By vectorizing $\mathbf{R}$, we obtain the length $N_R N$ vector $\mathbf{r} \stackrel{\triangle}{=} \text{vec}[\mathbf{R}] = \left[\mathbf{r}_1^{\mathbb{T}} \ldots \mathbf{r}_N^{\mathbb{T}}\right]^{\mathbb{T}}$, where the nth column $\mathbf{r}_n$ of $\mathbf{R}$ is the nth snapshot of the received signal. Similarly, $\mathbf{s}(\tau) = \text{vec}[\mathbf{S}(\tau)]$ and $\mathbf{w} = \text{vec}[\mathbf{W}]$ with covariance matrix $\mathbf{C}_w = \text{E}[\mathbf{w}\mathbf{w}^{\mathbb{H}}] = \mathbf{C}_T \otimes \mathbf{C}_S$. Here, $\mathbf{C}_T$ and $\mathbf{C}_S$ are the noise temporal and spatial covariances, $\mathbb{H}$ denotes conjugate transpose, $E[\cdot]$ denotes expectation, and $\otimes$ is the Kronecker product. The probability density function $p(\mathbf{r}|\boldsymbol{\psi})$ is a joint Gaussian density with mean $\mu(\boldsymbol{\psi}) = \text{vec}\left[\beta\,\mathbf{a}(\theta)\,\mathbf{v}^{\mathbb{T}}(\theta)\,\mathbf{S}(\tau)\,\mathbf{D}(\nu)\right]$ and covariance $\mathbf{C}_w$, where $\boldsymbol{\psi} = [\beta, \theta, \boldsymbol{\phi}]^{\mathbb{T}}$ and $\boldsymbol{\phi} = [\tau, \nu]^{\mathbb{T}}$. Using this formulation, our objective is to estimate $\boldsymbol{\psi}$ or, equivalently, the target range, range-rate, and DOA, where we treat the reflection coefficient β as a nuisance parameter.

B. Computation of CRLB and Transmit Waveform Parameters

The CRLB, $\text{CRLB}_{\boldsymbol{\psi}\boldsymbol{\psi}}$, on the covariance of the estimate of $\boldsymbol{\psi}$ can be obtained as the inverse of the Fisher information

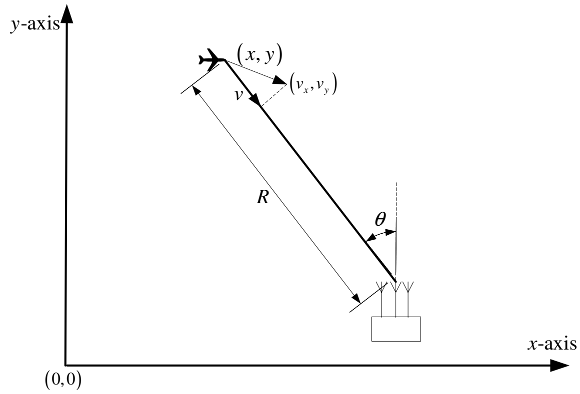

Fig. 1. MIMO radar system with colocated antennas.

matrix $\mathcal{I}(\boldsymbol{\psi})$ that can be computed as

$$
\begin{aligned}
\mathcal{I}(\boldsymbol{\psi}) &= 2\,\mathrm{Re}\Big\{\frac{\partial}{\partial\psi_i}\mu^{\mathbb{H}}(\boldsymbol{\psi})\ \mathbf{C}_w^{-1}\ \frac{\partial}{\partial\psi_j}\mu(\boldsymbol{\psi})\Big\} \\
&= \begin{bmatrix} \mathcal{I}_{\boldsymbol{\beta}\boldsymbol{\beta}} & \mathcal{I}_{\theta\beta}^{\mathbb{T}} & \mathcal{I}_{\boldsymbol{\phi}\beta}^{\mathbb{T}} \\ \mathcal{I}_{\theta\boldsymbol{\beta}} & \mathcal{I}_{\theta\theta} & \mathcal{I}_{\boldsymbol{\phi}\theta}^{\mathbb{T}} \\ \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\beta}} & \mathcal{I}_{\boldsymbol{\phi}\theta} & \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\phi}} \end{bmatrix}. \qquad (1)
\end{aligned}
$$

Here, ψ_i is the ith element of $\boldsymbol{\psi}$ and $\mathrm{Re}\{\cdot\}$ denotes the real part. The CRLB of $\theta\boldsymbol{\phi}$ can be obtained using

$$
\mathrm{CRLB}_{\theta\boldsymbol{\phi},\theta\boldsymbol{\phi}} = \left(\begin{bmatrix} \mathcal{I}_{\theta\theta} & \mathcal{I}_{\boldsymbol{\phi}\theta}^{\mathbb{T}} \\ \mathcal{I}_{\boldsymbol{\phi}\theta} & \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\phi}} \end{bmatrix} - \mathcal{J}\mathcal{I}_{\beta\beta}^{-1}\mathcal{J}^{\mathbb{T}}\right)^{-1}, \qquad (2)
$$

where $\mathcal{J} = [\mathcal{I}_{\theta\beta}^{\mathbb{T}}\ \ \mathcal{I}_{\boldsymbol{\phi}\beta}^{\mathbb{T}}]$ and $\mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\phi}} = \begin{bmatrix} \mathcal{I}_{\tau\tau} & \mathcal{I}_{\tau\nu} \\ \mathcal{I}_{\tau\nu}^{\mathbb{T}} & \mathcal{I}_{\nu\nu} \end{bmatrix}$. The computation of the Fisher information matrix can be found in [10]. Note that a similar derivation for the Fisher information matrix in (1) can be found in [14, Chapter 4] for the special case of white Gaussian noise and derived in the continuous time domain.

In order to obtain $\mathrm{CRLB}_{\boldsymbol{\psi}\boldsymbol{\psi}} = [\mathcal{I}(\boldsymbol{\psi})]^{-1}$, each element in (1) needs to be expressed in terms of β, $\mathbf{A}(\theta)$, $\frac{\partial}{\partial\theta}\mathbf{A}(\theta)$, $\mathbf{S}(\tau)$, $\frac{\partial}{\partial\tau}\mathbf{S}(\tau)$, $\mathbf{D}(\nu)$, and $\frac{\partial}{\partial\nu}\mathbf{D}(\nu)$. Although we cannot list here every element in (1), one example is

$$
\mathcal{I}_{\boldsymbol{\phi}\theta} = 2|\beta|^2\,\mathrm{Re}\left\{\begin{bmatrix} \mathrm{Tr}\Big\{\frac{\partial}{\partial\tau}\mathbf{S}^{\mathbb{H}}(\tau)\mathbf{A}^{\mathbb{H}}(\theta)\mathbf{C}_S^{-1}\frac{\partial}{\partial\theta}\mathbf{A}(\theta) \\ \cdot\mathbf{S}(\tau)\mathbf{D}(\nu)\mathbf{C}_T^{-1}\mathbf{D}^{\mathbb{H}}(\nu)\Big\} \\ \mathrm{Tr}\Big\{\mathbf{S}^{\mathbb{H}}(\tau)\mathbf{A}^{\mathbb{H}}(\theta)\mathbf{C}_S^{-1}\frac{\partial}{\partial\theta}\mathbf{A}(\theta) \\ \cdot\mathbf{S}(\tau)\mathbf{D}(\nu)\mathbf{C}_T^{-1}\frac{\partial}{\partial\nu}\mathbf{D}^{\mathbb{H}}(\nu)\Big\} \end{bmatrix}\right\}
$$

where $\mathrm{Tr}\{\cdot\}$ denotes matrix trace.

As our goal is to investigate the relationship between the CRLB in (2) and the transmitted waveform, we consider the following special case. For white temporal and spatial noise components, then $\mathbf{C}_T = \sigma_w^2\mathbf{I}_N$, $\mathbf{C}_S = \mathbf{I}_{N_R}$, where $\mathbf{I}_N$ is the $N\times N$ identity matrix. Then we can explicitly express the $\mathrm{CRLB}_{\theta\boldsymbol{\phi},\theta\boldsymbol{\phi}}$ in terms of the transmit waveform $\mathbf{S}(\tau)$. We first define $\mathbf{s}_{\theta\tau\nu} \triangleq \mathbf{v}^{\mathbb{T}}(\theta)\mathbf{S}(\tau)\mathbf{D}(\nu)$ as the delayed waveform $s_{\theta\tau\nu}(t) = \sum_{i=1}^{N_T} v_i(\theta)\,s_i(t-\tau)e^{j2\pi\nu t}$ from N_T antennas before discretization, where $v_i(\theta)$ is the ith element of $\mathbf{v}(\theta)$. Also, $\tilde{\mathbf{s}}_{\theta\tau\nu} \triangleq \frac{\partial}{\partial\theta}\mathbf{v}^{\mathbb{T}}(\theta)\mathbf{S}(\tau)\mathbf{D}(\nu)$, $\bar{\mathbf{s}}_{\theta\tau\nu} \triangleq \mathbf{v}^{\mathbb{T}}(\theta)\frac{\partial}{\partial\tau}\mathbf{S}(\tau)\mathbf{D}(\nu)$, $\breve{\mathbf{s}}_{\theta\tau\nu} \triangleq \mathbf{v}^{\mathbb{T}}(\theta)\mathbf{S}(\tau)\frac{\partial}{\partial\nu}\mathbf{D}(\nu) = j2\pi T_s\,\mathbf{s}_{\theta\tau\nu}\,\mathrm{diag}(1,2,\ldots,N)$, $\mathcal{S}_{\theta\tau\nu} \triangleq \mathbf{s}_{\theta\tau\nu}\mathbf{s}_{\theta\tau\nu}^{\mathbb{H}}$, $\tilde{\mathcal{S}}_{\theta\tau\nu} \triangleq \tilde{\mathbf{s}}_{\theta\tau\nu}\tilde{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\bar{\mathcal{S}}_{\theta\tau\nu} \triangleq \bar{\mathbf{s}}_{\theta\tau\nu}\bar{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\breve{\mathcal{S}}_{\theta\tau\nu} \triangleq \breve{\mathbf{s}}_{\theta\tau\nu}\breve{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\tilde{\mathscr{S}}_{\theta\tau\nu} \triangleq \tilde{\mathbf{s}}_{\theta\tau\nu}\mathbf{s}_{\theta\tau\nu}^{\mathbb{H}}$, $\bar{\mathscr{S}}_{\theta\tau\nu} \triangleq \mathbf{s}_{\theta\tau\nu}\bar{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\breve{\mathscr{S}}_{\theta\tau\nu} \triangleq \mathbf{s}_{\theta\tau\nu}\breve{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\bar{\mathscr{B}}_{\theta\tau\nu} \triangleq \bar{\mathbf{s}}_{\theta\tau\nu}\breve{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\tilde{\mathscr{B}}_{\theta\tau\nu} \triangleq \tilde{\mathbf{s}}_{\theta\tau\nu}\breve{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\mathscr{B}_{\theta\tau\nu} \triangleq \tilde{\mathbf{s}}_{\theta\tau\nu}\bar{\mathbf{s}}_{\theta\tau\nu}^{\mathbb{H}}$, $\mathcal{A}_\theta \triangleq \mathbf{a}(\theta)^{\mathbb{H}}\mathbf{a}(\theta)$, $\mathscr{A}_\theta \triangleq \frac{\partial}{\partial\theta}\mathbf{a}^{\mathbb{H}}(\theta)\,\frac{\partial}{\partial\theta}\mathbf{a}(\theta)$, and $\mathbb{A}_\theta \triangleq \frac{\partial}{\partial\theta}\mathbf{a}^{\mathbb{H}}(\theta)\,\mathbf{a}(\theta)$. Then, the elements in (1) can be written as

$$
\mathcal{I}_{\boldsymbol{\beta}\boldsymbol{\beta}} = \frac{2}{\sigma_w^2}\left(\mathcal{A}_\theta\,\mathcal{S}_{\theta\tau\nu}\right)\mathbf{I}_2,
$$

$$
\mathcal{I}_{\theta\theta} = \frac{2\,|\beta|^2}{\sigma_w^2}\mathrm{Re}\Big\{\mathscr{A}_\theta\,\mathcal{S}_{\theta\tau\nu} + \mathbb{A}_\theta\tilde{\mathscr{S}}_{\theta\tau\nu}^{\mathbb{H}} + \mathbb{A}_\theta^{\mathbb{H}}\tilde{\mathscr{S}}_{\theta\tau\nu} + \mathcal{A}_\theta\tilde{\mathcal{S}}_{\theta\tau\nu}\Big\},
$$

$$
\mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\phi}} = \frac{2\,|\beta|^2}{\sigma_w^2}\mathrm{Re}\left\{\begin{bmatrix} \mathcal{A}_\theta\bar{\mathcal{S}}_{\theta\tau\nu} & \mathcal{A}_\theta\bar{\mathscr{B}}_{\theta\tau\nu} \\ \mathcal{A}_\theta\bar{\mathscr{B}}_{\theta\tau\nu}^{\mathbb{H}} & \mathcal{A}_\theta\breve{\mathcal{S}}_{\theta\tau\nu} \end{bmatrix}\right\},
$$

$$
\mathcal{I}_{\theta\beta} = \frac{2}{\sigma_w^2}\mathrm{Re}\left\{[1\ j]\otimes\beta^*\left(\mathbb{A}_\theta\mathcal{S}_{\theta\tau\nu} + \mathcal{A}_\theta\tilde{\mathscr{S}}_{\theta\tau\nu}\right)\right\},
$$

$$
\mathcal{I}_{\boldsymbol{\phi}\beta} = \frac{2}{\sigma_w^2}\mathrm{Re}\left\{[1\ j]\otimes\begin{bmatrix} \beta^*\,\mathcal{A}_\theta\,\bar{\mathscr{S}}_{\theta\tau\nu} \\ \beta^*\,\mathcal{A}_\theta\,\breve{\mathscr{S}}_{\theta\tau\nu} \end{bmatrix}\right\},
$$

$$
\mathcal{I}_{\boldsymbol{\phi}\theta} = \frac{2|\beta|^2}{\sigma_w^2}\mathrm{Re}\left\{\begin{bmatrix} \mathbb{A}_\theta^{\mathbb{H}}\,\bar{\mathscr{S}}_{\theta\tau\nu} + \mathcal{A}_\theta\mathscr{B}_{\theta\tau\nu} \\ \mathbb{A}_\theta^{\mathbb{H}}\,\breve{\mathscr{S}}_{\theta\tau\nu} + \mathcal{A}_\theta\,\tilde{\mathscr{B}}_{\theta\tau\nu} \end{bmatrix}\right\}.
$$

If the transmission beam is steered toward the target, $\mathbf{v}(\theta) = \mathbf{1}$ and $\tilde{\mathbf{s}}_{\theta\tau\nu} = \frac{\partial}{\partial\theta}\mathbf{v}^{\mathbb{T}}(\theta)\,\mathbf{S}(\tau)\mathbf{D}(\nu) = \mathbf{0}$. Thus, the CRLB terms are decoupled and (2) becomes $\mathrm{CRLB}_{\theta\boldsymbol{\phi},\theta\boldsymbol{\phi}} = \mathrm{diag}(\mathrm{CRLB}_{\theta\theta}, \mathrm{CRLB}_{\boldsymbol{\phi}\boldsymbol{\phi}})$ where

$$
\mathrm{CRLB}_{\boldsymbol{\phi}\boldsymbol{\phi}} = \left(\mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\phi}} - \mathcal{I}_{\boldsymbol{\phi}\beta}\mathcal{I}_{\beta\beta}^{-1}\mathcal{I}_{\boldsymbol{\phi}\beta}^{\mathbb{T}}\right)^{-1} = (\Xi\,|\beta|^2\,N_R/\sigma_w^2)^{-1} \quad (3)
$$

$$
\begin{aligned}
\mathrm{CRLB}_{\theta\theta} &= \left(\mathcal{I}_{\theta\theta} - \mathcal{I}_{\theta\beta}\mathcal{I}_{\beta\beta}^{-1}\mathcal{I}_{\theta\beta}^{\mathbb{T}}\right)^{-1} \\
&= (\mathcal{S}_{\theta\tau\nu}\,2|\beta|^2/\sigma_w^2)^{-1}\left(\mathrm{Re}\Big\{\mathscr{A}_\theta - (\mathbb{A}_\theta\mathbb{A}_\theta^{\mathbb{H}})/(\mathcal{A}_\theta)\Big\}\right)^{-1} \quad (4)
\end{aligned}
$$

The CRLB of $\boldsymbol{\phi}$ is related to the transmitted waveform $s_{\theta\tau\nu}(t)$ and $s_{\theta\tau}(t) = \sum_{i=1}^{N_T} v_i(\theta)s_i(t-\tau)$. Specifically, the signal energy can be approximated by $\mathbf{s}_{\theta\tau\nu}\mathbf{s}_{\theta\tau\nu}^{\mathbb{H}} = \mathcal{S}_{\theta\tau\nu} \approx \int|s_{\theta\tau\nu}(t)|^2dt = E_s$, and the elements of Ξ in (3) are [10]

$$
\begin{aligned}
\xi_{1,1} &= \mathrm{Re}\left\{\bar{\mathcal{S}}_{\theta\tau\nu} - (\bar{\mathscr{S}}_{\theta\tau\nu}\bar{\mathscr{S}}_{\theta\tau\nu}^{\mathbb{H}})/\mathcal{S}_{\theta\tau\nu}\right\} \\
&\approx \int\left|\frac{\partial}{\partial t}s_{\theta\tau}(t)\right|^2 dt - \frac{1}{E_s}\left|\int s_{\theta\tau}(t)\Big(\frac{\partial}{\partial t}s_{\theta\tau}^*(t)\Big)dt\right|^2 \\
\xi_{1,2} &= \xi_{2,1} = \mathrm{Re}\left\{\bar{\mathscr{B}}_{\theta\tau\nu}^{\mathbb{H}} - (\breve{\mathscr{S}}_{\theta\tau\nu}\bar{\mathscr{S}}_{\theta\tau\nu}^{\mathbb{H}})/\mathcal{S}_{\theta\tau\nu}\right\} \\
&\approx 2\pi\ \mathrm{Im}\left\{\int t\,s_{\theta\tau\nu}(t)\Big(\frac{\partial}{\partial t}s_{\theta\tau}^*(t)\Big)dt\right. \\
&\quad \left. -\ \frac{1}{E_s}\int t\,|s_{\theta\tau}(t)|^2dt\int s_{\theta\tau}(t)\Big(\frac{\partial}{\partial t}s_{\theta\tau}^*(t)\Big)dt\right\} \\
\xi_{2,2} &= \mathrm{Re}\left\{\breve{\mathcal{S}}_{\theta\tau\nu} - (\breve{\mathscr{S}}_{\theta\tau\nu}^{\mathbb{H}}\breve{\mathscr{S}}_{\theta\tau\nu})/\mathcal{S}_{\theta\tau\nu}\right\} \\
&\approx 4\pi^2\int t^2\ |s_{\theta\tau\nu}(t)|^2\,dt - \frac{4\pi^2}{E_s}\left|\int t\,|s_{\theta\tau}(t)|^2dt\right|^2.
\end{aligned}
$$

The terms $\xi_{1,1}$ and $\xi_{2,2}$ are proportional to the root mean-squared (rms) bandwidth and rms duration of $s_{\theta\tau\nu}(t)$, respectively. Also, the Ξ elements in (3) are invariant to translation in time and frequency [15].

C. Comparison between MIMO and Phased Array Radars

Phased array radars can be treated as a subset of MIMO radars with colocated antennas as the antennas transmit the same signal but with different phase. In [16], the authors argued that MIMO radars suffer from a substantial loss in SNR relative to phased array radars when orthogonal waveforms are used. Although this can be demonstrated using the derived CRLB, MIMO radars are not restricted to orthogonal waveforms, and we provide a numerical example to demonstrate how MIMO radars can use waveform design to decouple range-Doppler estimation.

We start by showing the loss in SNR when orthogonal waveforms are used with MIMO radars. The first term in (4), $2\mathcal{S}_{\theta\tau\nu}|\beta|^2/\sigma_w^2$ represents the gain in SNR. We assume that the beam pattern is already steered toward the target, i.e., $\mathbf{v}(\theta) = \mathbf{1}$. For a phased array radar, $s_{\theta\tau\nu}(t) = \sum_{i=1}^{N_T} 1 \cdot s(t-\tau)e^{j2\pi\nu t} = N_T s(t-\tau)e^{j2\pi\nu t}$, where $s(t)$ is the same waveform transmitted by all the antennas in the phased array radar. As a result, $2\mathcal{S}_{\theta\tau\nu}|\beta|^2/\sigma_w^2 = 2N_T^2|\beta|^2 E_s/\sigma_w^2$, where $E_s = \int |s(t)|^2 dt$ is the energy of the waveform. For a MIMO radar, if we assume the waveforms $s_i(t)$, $i = 1, 2, \cdots, N_T$, are *orthogonal* to each other, and the transmission energy for each waveform is equal and $E_s = \int |s_i(t)|^2 dt$, then $s_{\theta\tau\nu}(t) = \sum_{i=1}^{N_T} 1 \cdot s_i(t-\tau)e^{j2\pi\nu t}$ and $\mathcal{S}_{\theta\tau\nu} = \int |s_{\theta\tau\nu}(t)|^2 dt = N_T E_s$. Then, the SNR gain is $2\mathcal{S}_{\theta\tau\nu}|\beta|^2/\sigma_w^2 = 2N_T|\beta|^2 E_s/\sigma_w^2$. Thus, the SNR gain for the phase array radar is N_T times higher than the SNR gain for the MIMO radar; this is for the CRLB of the DOA estimation when orthogonal waveforms are used by the MIMO radar.

We would like to point out that this SNR loss is only valid under the orthogonal waveform assumption. What enables the phased array radars to achieve the antenna gain is the coherent combining of the signals at each antenna by adjusting the phase of each return so that the phases align. The orthogonal waveforms are one mechanism through which we cannot coherently combine the signals. MIMO radars do not require waveforms to be orthogonal. This means that SNR gains through coherent combining of the waveforms is possible. Even when orthogonal waveforms are used, the range-Doppler estimation may still outperform phased array radar due to MIMO radar's ability of decoupling range-Doppler estimation by transmitting different waveforms at different antennas, as demonstrated next.

We compared the CRLB for estimating target attributes using a phased array radar and a MIMO radar with $N_T = 3$ transmit and $N_R = 3$ receive colocated antennas. The following parameters were used in the simulations. The speed of sound in the air was $c = 3 \times 10^8$ m/s, the carrier frequency was $f_c = 1$ GHz (thus the wavelength was $\lambda = 0.3$ m), $\theta = \pi/6$, the target range was 10 km (thus $\tau = 67$ μs) and $\nu = 1$ kHz. The distance between transmit antennas (and also between receiver antennas) was 0.5λ.

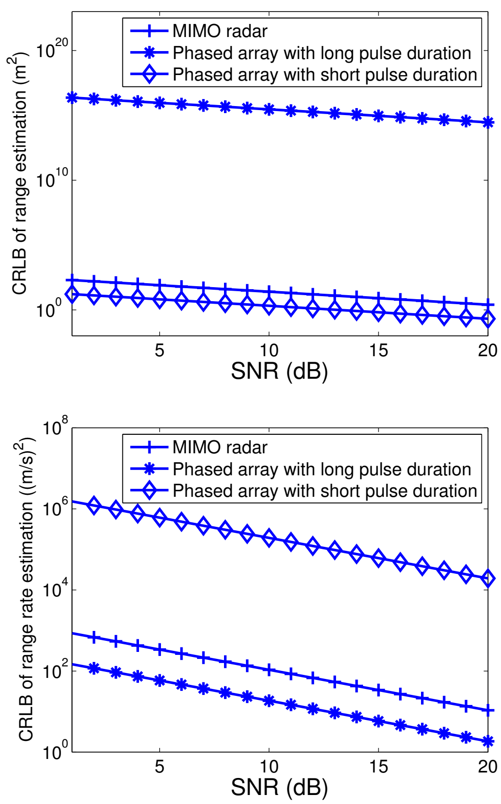

Fig. 2. CRLB for estimating range (top) and range-rate (bottom).

Figure 2 compares the CRLB of a phased array radar and a MIMO radar system under different waveform parameter configurations. For the phased array radar, the three antennas were assumed to transmit the same duration pulse. We considered two different cases: 1 ms and 0.01 ms duration pulses. For MIMO radar, we assume that the three antennas transmit different pulses with duration 1 ms, 0.1 ms and 0.01 ms, respectively. As we can see, when the phased array radar transmits the long duration pulse, the range-rate estimation for the phased array radar is better than for the MIMO radar; however, the CRLB for the range estimation is significantly deteriorated. On the other hand, when the phased array radar transmits the short duration pulse, the reverse is true. As a result, the MIMO radar obtains the near-optimal joint estimation of both range and range-rate.

III. CRLB Using Widely-Separated Antenna MIMO Radars

A. Signal and Noise Models

We consider a MIMO radar system with widely-separated antennas as depicted in Figure 3. The signal transmitted by the mth antenna, $m = 1, 2, \ldots, N_T$, and received by the lth antenna, $l = 1, 2, \ldots, N_R$, can be represented by

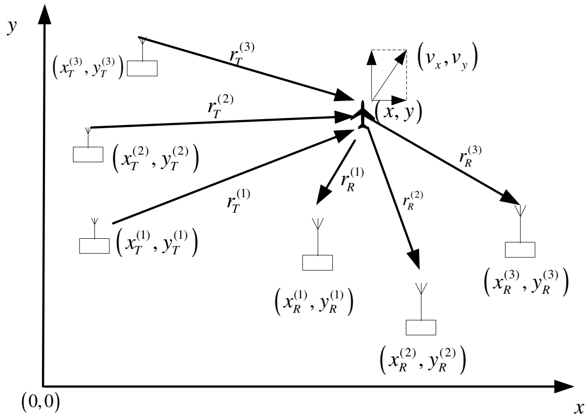

Fig. 3. MIMO radar system with widely-separated antennas.

$r_{l,m}(t) = \beta_{l,m} s_{l,m}(t) + w_{l,m}(t)$. If $s_m(t)$ is the signal transmitted by the mth antenna with carrier frequency f_m, then $s_{l,m}(t) = s_m(t - \tau_{l,m}) e^{j2\pi\nu_{l,m} t}$. The reflection coefficients, time-delay, and Doppler-shift between the (l,m)th radar pair are given by $\beta_{l,m}$, $\tau_{l,m}$, and $\nu_{l,k}$, respectively. We assume that $w_{l,m}(t)$ is additive white Gaussian noise (AWGN) with $\mathrm{E}[w_{l,m}(t) w_{l',m'}(t)] = 0$, $l \neq l'$, $m \neq m'$, where $\mathrm{E}[\cdot]$ denotes statistical expectation. After sampling with period T_s, the signal samples are given by $r_{l,m}[n] = r_{l,m}(nT_s)$, $n = 1, \ldots, N$. We form the received signal vector $\mathbf{r} = [\mathbf{r}_1^{\mathbb{T}}, \cdots, \mathbf{r}_{N_T}^{\mathbb{T}}]^{\mathbb{T}}$, where $\mathbf{r}_m = [\mathbf{r}_{1,m}^{\mathbb{T}}, \cdots, \mathbf{r}_{N_R,m}^{\mathbb{T}}]^{\mathbb{T}}$, $\mathbf{r}_{l,m} = [r_{l,m}[1], \cdots, r_{l,m}[N]]^{\mathbb{T}}$, and $\mathbb{T}$ denotes vector transpose. Similarly, we define $\mathbf{s}_{l,m} = [s_{l,m}[1], \cdots, s_{l,m}[N]]^{\mathbb{T}}$ and $\mathbf{w}_{l,m} = [w_{l,m}[1], \cdots, w_{l,m}[N]]^{\mathbb{T}}$. Also, $\boldsymbol{\beta} = [\boldsymbol{\beta}_{1,1}^{\mathbb{T}} \cdots \boldsymbol{\beta}_{l,m}^{\mathbb{T}} \cdots \boldsymbol{\beta}_{N_R,N_T}^{\mathbb{T}}]^{\mathbb{T}}$, where $\boldsymbol{\beta}_{l,m} = [\mathrm{Re}\{\beta_{l,m}\} \;\; \mathrm{Im}\{\beta_{l,m}\}]^{\mathbb{T}}$.

B. CRLB Computation

Using the above signal model, we obtain the probability density function of the received signal as

$$p(\mathbf{r}|\boldsymbol{\beta}, x, y, v_x, v_y) = (\pi^{N N_R N_T} \det(\mathbf{C}_w))^{-1} \cdot \exp\left(-\sum_{m=1}^{N_T} \sum_{l=1}^{N_R} \left[(\mathbf{r}_{l,m} - \beta_{l,m}\mathbf{s}_{l,m})^{\mathbb{H}} \mathbf{C}_w^{-1} (\mathbf{r}_{l,m} - \beta_{l,m}\mathbf{s}_{l,m}) \right] \right),$$

where (x, y) and (v_x, v_y) are the position and velocity of the target in two-dimensional Cartesian coordinates, $\mathbf{C}_w$ is the covariance matrix of $\mathbf{w}_{l,m}$, $\det(\cdot)$ denotes matrix determinant, and $\mathbb{H}$ denotes vector Hermitian. With the AWGN assumption, the noise covariance is given by $\mathbf{C}_w = \sigma_w^2 \mathbf{I}_N$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix. We let the unknown parameter vector be given by $\boldsymbol{\Psi} = [x, y, v_x, v_y]^{\mathbb{T}}$. Then, the CRLB for estimating $\boldsymbol{\Psi}$ is given by [10]

$$\mathrm{CRLB}_{\boldsymbol{\Psi}\boldsymbol{\Psi}} = \sigma_w^2 \left[2 \sum_{m=1}^{N_T} \sum_{l=1}^{N_R} |\beta_{l,m}|^2 \mathbf{H}_{l,m}^{\mathbb{T}} \, \mathcal{I}_{l,m}^{\boldsymbol{\Phi}} \, \mathbf{H}_{l,m} \right]^{-1} \quad (5)$$

where

$$\mathcal{I}_{l,m}^{\boldsymbol{\Phi}} = \mathrm{Re}\left\{ \left(\frac{\partial \mathbf{s}_{l,m}}{\partial \boldsymbol{\Phi}_{l,m}^{\mathbb{T}}} \right)^{\mathbb{H}} \frac{\partial \mathbf{s}_{l,m}}{\partial \boldsymbol{\Phi}_{l,m}^{\mathbb{T}}} - \left(\frac{\partial \mathbf{s}_{l,m}}{\partial \boldsymbol{\Phi}_{l,m}^{\mathbb{T}}} \right)^{\mathbb{H}} \mathbf{s}_{l,m} (\mathbf{s}_{l,m}^{\mathbb{H}} \mathbf{s}_{l,m})^{-1} \mathbf{s}_{l,m}^{\mathbb{H}} \frac{\partial \mathbf{s}_{l,m}}{\partial \boldsymbol{\Phi}_{l,m}^{\mathbb{T}}} \right\} \quad (6)$$

is the waveform characteristic matrix, and

$$\mathbf{H}_{l,m} = \frac{\partial \boldsymbol{\Phi}_{l,m}}{\partial \boldsymbol{\Psi}^{\mathbb{T}}} = \begin{bmatrix} \frac{\partial \tau_{l,m}}{\partial x} & \frac{\partial \tau_{l,m}}{\partial y} & \frac{\partial \tau_{l,m}}{\partial v_x} & \frac{\partial \tau_{l,m}}{\partial v_y} \\ \frac{\partial \nu_{l,m}}{\partial x} & \frac{\partial \nu_{l,m}}{\partial y} & \frac{\partial \nu_{l,m}}{\partial v_x} & \frac{\partial \nu_{l,m}}{\partial v_y} \end{bmatrix}, \quad (7)$$

which can be determined by the radar geometry. Specifically, if the mth transmission antenna of a MIMO radar system is placed at $(x_T^{(m)}, y_T^{(m)})$, $m = 1, \ldots, N_T$, and the lth receiver antenna is placed at $(x_R^{(l)}, y_R^{(l)})$, $l = 1, \ldots, N_R$, then the time delay $\tau_{l,m}$ and Doppler shift $\nu_{l,m}$ are given by

$$\tau_{l,m} = \frac{r_T^{(m)} + r_R^{(l)}}{c} \text{ and } \nu_{l,m} = \frac{f_m \left(\dot{r}_T^{(m)} + \dot{r}_R^{(l)} \right)}{c}$$

where $r_T^{(m)} = \left((x - x_T^{(m)})^2 + (y - y_T^{(m)})^2 \right)^{1/2}$, $r_R^{(l)} = \left((x - x_R^{(l)})^2 + (y - y_R^{(l)})^2 \right)^{1/2}$, $\dot{r}_T^{(m)} = (v_x(x - x_T^{(m)}) + v_y(y - y_T^{(m)}))/r_T^{(m)}$, $\dot{r}_R^{(l)} = (v_x(x - x_R^{(l)}) + v_y(y - y_R^{(l)}))/r_R^{(l)}$. As a result, the elements of $\mathbf{H}_{l,m}$ are given by [17]

$$\frac{\partial \tau_{l,m}}{\partial x} = \frac{1}{c} \left(\frac{x - x_T^{(m)}}{r_T^{(m)}} + \frac{x - x_R^{(l)}}{r_R^{(l)}} \right)$$

$$\frac{\partial \tau_{l,m}}{\partial y} = \frac{1}{c} \left(\frac{y - y_T^{(m)}}{r_T^{(m)}} + \frac{y - y_R^{(l)}}{r_R^{(l)}} \right)$$

$$\frac{\partial \tau_{l,m}}{\partial v_x} = \frac{\partial \tau_{l,m}}{\partial v_y} = 0$$

$$\frac{\partial \nu_{l,m}}{\partial x} = \frac{f_m}{c} \left(\frac{v_x}{r_T^{(m)}} - \dot{r}_T^k \frac{x - x_T^{(m)}}{(r_T^{(m)})^2} + \frac{v_x}{r_R^{(l)}} - \dot{r}_R^l \frac{x - x_R^{(l)}}{(r_R^{(l)})^2} \right)$$

$$\frac{\partial \nu_{l,m}}{\partial y} = \frac{f_m}{c} \left(\frac{v_y}{r_T^{(m)}} - \dot{r}_T^k \frac{y - y_T^{(m)}}{(r_T^{(m)})^2} + \frac{v_y}{r_R^{(l)}} - \dot{r}_R^l \frac{y - y_R^{(l)}}{(r_R^{(l)})^2} \right)$$

$$\frac{\partial \nu_{l,m}}{\partial v_x} = \frac{f_m}{c} \left(\frac{x - x_T^{(m)}}{r_T^{(m)}} + \frac{x - x_R^{(l)}}{r_R^{(l)}} \right)$$

$$\frac{\partial \nu_{l,m}}{\partial v_y} = \frac{f_m}{c} \left(\frac{y - y_T^{(m)}}{r_T^{(m)}} + \frac{y - y_R^{(l)}}{r_R^{(l)}} \right)$$

where c is the velocity in the propagation medium.

C. Comparison Between MIMO Radar and Multistatic Radar

For multistatic radar, we use the same antenna configuration and signal model as for MIMO radar in Section III-A. The difference between multistatic radar and MIMO radar with widely-separated antennas is stated as follows. The multistatic radar first obtains range and Doppler estimates $\mathbf{\Phi}_{l,m} = [\tau_{l,m}\ \nu_{l,m}]^{\mathbb{T}}$ for each radar pair (l,m), $l = 1,2,\ldots,N_R$, $m = 1,2,\ldots,N_T$, and then it uses these estimates to obtain the target's location and velocity $\mathbf{\Psi} = [x\ y\ v_x\ v_y]^{\mathbb{T}}$. Next, we derive the CRLB on estimating $\mathbf{\Psi}$ using multistatic radar and maximum likelihood estimation. We let $\widehat{\mathbf{\Phi}}_{l,m} = \mathbf{\Phi}_{l,m} + \mathbf{v}_{l,m}$ be the estimate of $\mathbf{\Phi}_{l,m}$, and $\mathbf{v}_{l,m}$ is additive Gaussian noise with covariance $\mathbf{C}^{\mathbf{\Phi}}_{l,m}$. We also concatenate $\widehat{\mathbf{\Phi}}_{l,m}$ into vector $\widehat{\mathbf{\Phi}} = [\widehat{\mathbf{\Phi}}^{\mathbb{T}}_{1,1} \cdots \widehat{\mathbf{\Phi}}^{\mathbb{T}}_{N_R,N_T}]^{\mathbb{T}}$. Using this notation, the conditional probability density of $\widehat{\mathbf{\Phi}}$ given $\mathbf{\Psi}$ is

$$p(\widehat{\mathbf{\Phi}}|\mathbf{\Psi}) = \left((2\pi)^{2N_R N_T} \prod_{l=1}^{N_R} \prod_{m=1}^{N_T} \det(\mathbf{C}^{\mathbf{\Phi}}_{l,m}) \right)^{-\frac{1}{2}}$$

$$\cdot \exp\left(-\frac{1}{2} \sum_{l=1}^{N_R} \sum_{m=1}^{N_T} (\widehat{\mathbf{\Phi}}_{l,m} - \mathbf{\Phi}_{l,m})^{\mathbb{T}} \mathbf{C}^{\mathbf{\Phi}-1}_{l,m} (\widehat{\mathbf{\Phi}}_{l,m} - \mathbf{\Phi}_{l,m}) \right). \quad (8)$$

Based on (8), the Fisher information matrix $\mathcal{I}_{\text{ms}}(\mathbf{\Psi})$ for estimating $\mathbf{\Psi}$ can be obtained as

$$\begin{aligned} \mathcal{I}_{\text{ms}}(\mathbf{\Psi}) &= \sum_{l=1}^{N_R} \sum_{m=1}^{N_T} \left(\frac{\partial}{\partial \mathbf{\Psi}^{\mathbb{T}}} \mathbf{\Phi}_{l,m} \right)^{\mathbb{T}} \mathbf{C}^{\mathbf{\Phi}-1}_{l,m} \frac{\partial}{\partial \mathbf{\Psi}^{\mathbb{T}}} \mathbf{\Phi}_{l,m} \\ &= \sum_{l=1}^{N_R} \sum_{m=1}^{N_T} \mathbf{H}^{\mathbb{T}}_{l,m} \mathbf{C}^{\mathbf{\Phi}-1}_{l,m} \mathbf{H}_{l,m}\ , \end{aligned} \quad (9)$$

and the CRLB of estimating $\mathbf{\Phi}$ using multistatic radar is

$$\text{CRLB}_{\text{ms}} = \mathcal{I}^{-1}_{\text{ms}}(\mathbf{\Psi})\,.$$

Comparing (9) with (5), we notice that the CRLBs of multistatic radar and MIMO radar have the same form. We also notice that the CRLB corresponding to the radar pair (l,m) for estimating $\mathbf{\Phi}_{l,m}$ is $\text{CRLB}^{\mathbf{\Phi}}_{l,m} = (2|\beta_{l,m}|^2 \mathcal{I}^{\mathbf{\Phi}}_{l,m})^{-1}$. As a result, if we assume that the covariance of $\widehat{\mathbf{\Phi}}_{l,m}$ for all radar pairs approaches the CRLB, then multistatic radars and MIMO radars with widely-separated antennas will have the same estimation performance bound. However, the assumption that all radar pairs have covariances that approach the CRLB is difficult to achieve. On the other hand, for MIMO radar, due to their processing gain, they can achieve the CRLB; the snapshot data would need to first be sent to the processing center and a lot of data would need to be transmitted.

IV. Conclusion

In this paper, we derived the CRLB for jointly estimating parameters of moving targets using MIMO radar systems. We considered both MIMO radars with colocated antennas and widely-separated antennas. We specifically compared the CRLB of colocated-antenna MIMO radars with that of phased array radars, and the CRLB of widely-separated antenna MIMO radars with that of multistatic radars. When comparing MIMO radar configurations with corresponding radar array configurations, we showed that the MIMO radar does not suffer from trade-offs in jointly estimating target attributes since different antennas can transmit different waveforms.

References

[1] J. Li and P. Stoica, "MIMO radar with colocated antennas," *IEEE Signal Processing Magazine*, vol. 24, pp. 106–114, 2007.

[2] J. Li, L. Xu, P. Stoica, K. W. Forsythe, and D. W. Bliss, "Range compression and waveform optimization for MIMO radar: A Cramér Rao bound based study," *IEEE Transactions on Signal Processing*, vol. 56, no. 1, pp. 218–232, 2008.

[3] L. Xu and J. Li, "Iterative generalized-likelihood ratio test for MIMO radar," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2375–2385, 2007.

[4] D. R. Fuhrmann and G. S. Antonio, "Transmit beamforming for MIMO radar systems using partial signal correlation," in *Asilomar Conference on Signals, Systems and Computers*, vol. 1, 2004, pp. 295–299.

[5] T. Aittomäki and V. Koivunen, "Low-complexity method for transmit beamforming in MIMO radars," in *IEEE International Conference on Acoustics, Speech and Signal Processing*, vol. 2, 2007, pp. 305–308.

[6] E. Fishler, A. Haimovich, R. Blum, D. Chizhik, L. Cimini, and R. Valenzuela, "MIMO radar: An idea whose time has come," in *IEEE Radar Conf.*, 2004, pp. 71–78.

[7] E. Fishler, A. Haimovich, R. Blum, L. J. Cimini, Jr., D. Chizhik, and R. Valenzuela, "Spatial diversity in radars-models and detection performance," *IEEE Trans. on Signal Processing*, vol. 54, no. 3, pp. 823–838, 2006.

[8] N. H. Lehmann, E. Fishler, A. M. Haimovich, R. S. Blum, D. Chizhik, L. J. Cimini, and R. A. Valenzuela, "Evaluation of transmit diversity in MIMO-radar direction finding," *IEEE Trans. on Signal Processing*, vol. 55, pp. 2215–2225, 2007.

[9] A. M. Haimovich, R. S. Blum, and L. J. Cimini, "MIMO radar with widely separated antennas," *IEEE Signal Processing Mag.*, vol. 25, no. 1, pp. 116–129, 2008.

[10] J. Zhang, "Waveform diversity and design for agile sensing and environment characterization," Ph.D. dissertation, Arizona State University, Tempe, AZ, December 2008.

[11] J. Zhang, B. Manjunath, G. Maalouli, A. Papandreou-Suppappola, and D. Morrell, "Dynamic waveform design for target tracking using MIMO radar," in *Asilomar Conference on Signals, Systems and Computers*, October 2008.

[12] G. San Antonio, D. R. Fuhrmann, and F. C. Robey, "MIMO radar ambiguity functions," *IEEE Journal of Selected Topics in Signal Processing*, vol. 1, no. 1, pp. 167–177, 2007.

[13] N. H. Lehmann, A. M. Haimovich, R. S. Blum, and L. Cimini, "High resolution capabilities of MIMO radar," in *Asilomar Conf. on Signals, Systems and Computers*, 2006, pp. 25–30.

[14] J. Li and P. Stoica, Eds., *MIMO radar signal processing*. Wiley-IEEE Press, October 2008.

[15] A. Dogandzic and A. Nehorai, "Cramér-Rao bounds for estimating range, velocity, and direction with a sensor array," *IEEE Trans. on Signal Processing*, vol. 49, pp. 1122–1137, 2001.

[16] F. Daum and J. Huang, "MIMO radar: snake oil or good idea?" October 2008.

[17] B. Ristic and S. Arulampalam, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*. Artech House, 2004.