Manifold Optimization for Joint Design of MIMO-STAP Radars

Jie Li †, Student Member, IEEE, Guisheng Liao ‡, Senior Member, IEEE, Yan Huang †, Member, IEEE, and Arye Nehorai †, Life Fellow, IEEE

Abstract—In order to maximize the signal-to-interference-plus-noise ratio (SINR) under a constant-envelope (CE) constraint, a fast and efficient joint design of the transmit waveform and the receive filter for colocated multiple-input multiple-output (MIMO) radars is essential. Conventional joint optimization is performed using nonlinear optimization techniques such as the semidefinite relaxation (SDR) algorithm. In this letter, we propose a novel manifold-based alternating optimization (MAO) method, which reformulates the waveform optimization subproblem as an unconstrained optimization problem on a Riemannian manifold. We present the geometrical structure of the feasible region and derive the explicit expressions for the Riemannian gradient and the Riemannian Hessian, thus the reformulated optimization could be solved by using the Riemannian trust-region (RTR) algorithm. Numerical experiments demonstrate that the proposed method has faster convergence with reduced computational cost compared with conventional SDR-based algorithm in Euclidean space.

Index Terms—Multiple-input multiple-output (MIMO) radar, manifold optimization, waveform design, receive filter design.

I. INTRODUCTION

MULTIPLE-INPUT multiple-output (MIMO) radar has been widely studied over the past decades [1]. For both colocated MIMO radars [2] and distributed MIMO radars [3], one of the most concerning problems is how to design probing signals appropriately. Existing approaches can be largely classified into five categories: optimizing an estimation–oriented lower bound [4], optimizing the radar ambiguity function [5], achieving a desired beampattern [6], optimizing the detection performance based on information theory [7], and joint design of transmitter and receiver to maximize the signal-to-interference-plus-noise ratio (SINR) [8]. Moreover, it is known that the space-time adaptive processing (STAP) technology has a strong ability to suppress interference [9]. Compared with conventional single-input multiple-output (SIMO) STAP systems, MIMO-STAP systems have much sharper clutter notches and improved minimum detectable velocity (MDV) performance [10].

In this letter, we focus on the joint design of transmit waveform and receive filter for airborne colocated MIMO-STAP radars to improve the detection performance of the low-velocity moving target. Counting in the constant-envelope (CE) constraint, the joint optimization problem is an intractable non-convex optimization problem. Most of the existing approaches for solving the interested joint optimization problem use the semidefinite relaxation (SDR) followed by Gaussian randomization [11]. Whereas, the SDR-based algorithm is not scalable to large-scale antenna systems since the number of variables is quadratically dependent on the number of antennas. In addition, extracting a rank-one component from the optimum solution to the SDR problem is NP-hard in general.

The development of optimizations on matrix manifolds has burgeoned during the past decade [12]. When provided with underlying geometries of the feasible sets, using the geometrical structures can deliver high quality solutions to the NP-hard problem with much lower computational cost. Hence, by a fully consideration of the abovementioned difficulties, we propose a novel manifold-based alternating optimization (MAO) algorithm which tackles the optimization problem directly on a Riemannian manifold. We view the feasible sets, which are Hermitian, positive semidefinite (SDP), and fixed-rank matrices with a unit diagonal, as a Riemannian manifold. Then the geometrical structure of the feasible sets is established and the waveform optimization problem is reformulated as an unconstrained problem on this manifold. Next, we strictly derive the explicit expressions for the Riemannian gradient and the Riemannian Hessian. Thus, the Riemannian trust-region (RTR) method [12] can be developed to solve the optimization problem instead of using the algorithms on Euclidean space. As a result, the proposed algorithm achieves better performance and avoids the time-consuming randomization procedure.

Notations: \{·\}T, \{·\}∗, and \{·\}H denote, respectively, transpose, conjugate, and conjugate transpose. vec(·) stacks the columns of a matrix to construct a vector. tr(·) represents the trace of a matrix. IN denotes an identity matrix of size N × N. 0N denotes the N-dimensional zero vector. diag(·) gets the diagonal elements of A to construct a vector. i indicates the all-ones vector. |a| denotes the modulus of a. ⊗ and ⊙ denote, respectively, the Kronecker product and the Hadamard product. \Re{·} indicates the element-wise real part of z.
II. SIGNAL MODEL AND PROBLEM FORMULATION

Consider an airborne colocated MIMO radar with $N_t$ transmitting antennas and $N_r$ receiving antennas. Both of the transmitting and receiving arrays are equispaced with inter-element spacings $d_t$ and $d_r$, respectively. The radar transmits a burst of $K$ pulses in a coherent processing interval (CPI) at a constant pulse repetition frequency (PRF) $f_r$. We assume that the waveform of each transmit antenna is repeated pulse to pulse. Let $S \in \mathbb{C}^{N_t \times L}$ denote waveform matrix of the $k$-th pulse with $L$ being the code length. At each receiver, the received signal is down-converted to baseband, and the received signal of a far-field moving target for the $k$-th pulse is given by

$$ Y_{T,k} = \alpha_T e^{j2\pi(k-1)f_{TR}t} b(\psi_T) a^T(\psi_T) S, $$

(1)

where $\alpha_T$ is the target complex amplitude, $a(\psi_T)$ and $b(\psi_T)$ denote, respectively, the transmit and the receive spatial steering vectors of the target, $\lambda$ is the radar wavelength, $\psi_T$ is the direction of angle (DOA) of the target, satisfied $\cos(\psi_T) = \cos(\phi_T) \sin(\theta_T)$ with $\phi_T$ and $\theta_T$ being the elevation and the azimuth of the target, and $f_{TR}$ is the normalized target Doppler frequency. Stacking all the columns of $Y_{T,k}$, we have

$$ y_T = \alpha_T (I_{N_r, K} \otimes S^T) v_T, $$

(2)

where $v_T = u(f_T) \otimes b(\psi_T) \otimes a(\psi_T)$ denotes the virtual steering vector of the target with $u(f_T)$ being the temporal steering vector of the target. Consider that the signal-dependent interference is the superposition of the echoes from different uncorrelated scatterers [13], as illustrated in Fig. 1, from a number of range rings. Specifically, if the target is at the $r$-th range ring, the received interference vector associated with the $[r-P, r+P]$ range cells is modeled by

$$ y_I = \sum_{p=-P}^{P} \sum_{q=1}^{N_r} \alpha_{I,p,q} (I_{N_r, K} \otimes J_p^T S^T) v_{I,p,q}, $$

(3)

where $N_r$ is the number of discrete azimuth sectors in the interested range ring, $v_{I,p,q} = u(f_{I,p,q}) \otimes b(\psi_{I,p,q}) \otimes a(\psi_{I,p,q})$, and $\alpha_{I,p,q}$ represent, respectively, the virtual steering vector, and the complex amplitude of the $(r+p,q)$-th interference source. $v_{I,p,q}$ and $f_{I,p,q}$ denote, respectively, the DOA and the normalized Doppler frequency of the $(r+p,q)$-th interference. $J_p \in \mathbb{R}^{L \times L}$ is the shift matrix [11]. Note that the target and interference information can be obtained from a prior knowledge by suing an environment dynamic database [14]. The noise vector $n \in \mathbb{C}^{N_r, L \times 1}$ is circularly symmetric complex Gaussian with zero mean and covariance $R_n = \sigma_n^2 I_{N_r, K, L}$, where $\sigma_n^2$ is the power of the noise. Herein, we assume that $\sigma_n^2 = 1$. Thus, the entire received signals $y$ within a CPI can be recast as

$$ y = y_T + y_I + n. $$

(4)

In order to maximize the output signal-to-interference-plus-noise ratio (SINR), we formalize the joint optimization of the transmit waveform and the receive filter under a practical constraint. Assuming that the observations $y$ is filtered through a linear finite impulse response filter $w \in \mathbb{C}^{N_t, K \times 1}$, the SINR can be expressed as

$$ \text{SINR}(w, S) = \frac{\|w^H S^T v_T^H S^T w\|}{w^H (\Phi + R_n) w}, $$

(5)

where $\Phi = \sum_{p=-P}^{P} \sum_{q=1}^{N_r} \sigma_{I,p,q}^2 S_p^T v_{I,p,q} v_{I,p,q}^H S_p^*$ is the interference covariance matrix with $\sigma_{I,p,q}^2$ being the variance of the $(r+p,q)$-th interference, $S = I_{N_t, K} \otimes S$, and $S_p = I_{N_r, K} \otimes S J_p$.

In practical applications, the synthesized waveform should be constant modulus due to the restriction of the nonlinear radar amplifiers [15]. Thus, the constant-envelope (CE) constraint is incorporated in the joint optimization problem, which can be formulated as

$$ \max_{w, R} \quad \text{SINR}(w, R) = \frac{\|w^H \Theta w\|}{\|w^H (\Phi + R_n) w\|} = \frac{\text{tr}(\Gamma R)}{\text{tr}(\Sigma R)}, $$

(6)

s.t. $\text{diag}(R) = \frac{P_0}{N_t L} \cdot 1$, rank($R$) = 1, $R \succeq 0$.

where

$$ \Theta = (\Upsilon_T \otimes \Lambda_T) (A_T P_T R^* P A_T^H), $$

(7)

$$ \Phi = \sum_{p=-P}^{P} \sum_{q=1}^{N_r} \sigma_{I,p,q}^2 (\Upsilon_{I,p,q} \otimes \Lambda_{I,p,q}) $$

$$ \otimes (A_{I,p,q} P_T R^* P A_{I,p,q}^H), $$

(8)

$$ \Gamma = (W^* \otimes I_{N_t}) v_T v_T^H (W^T \otimes I_{N_t}), $$

(9)

$$ \Sigma = \sum_{p=-P}^{P} \sum_{q=1}^{N_r} \sigma_{I,p,q}^2 [J_p W^* (\Upsilon_{I,p,q} \otimes \Lambda_{I,p,q}) W^T J_{-p} ] $$

$$ \otimes (a(\psi_{I,p,q}) a^H (\psi_{I,p,q})), $$

(10)

with $\Upsilon_T = u_T u_T^H$, $\Upsilon_{I,p,q} = u_{I,p,q} u_{I,p,q}^H$, $A_T = a_T^T (\psi_T) \otimes I_L$, $A_{I,p,q} = a_T^T (\psi_{I,p,q}) \otimes J_{-p}$, $A_T = b_T b_T^H$, $A_{I,p,q} = b_{I,p,q} b_{I,p,q}^H$, $P$ being a communication matrix such that $\text{vec}(S) = P (\text{vec}(S^T))$, $R = s^* s^T$ is the waveform covariance matrix (WCM) with $s = \text{vec}(S)$. $P_0$ is the total transmit power of $N_t$ transmitters. Note that the CE constraint of waveform $S$ is transformed to the rank constraint of the WCM $R$. Problem (6) is NP-hard in general [16], which has no close-form solution. Herein, we propose a novel manifold-based alternating optimization (AOO) algorithm which provides high quality solutions to the NP-hard problem (6).

III. MANIFOLD-BASED ALTERNATING OPTIMIZATION METHOD

In this section, we leverage the alternating optimization to sequentially increase the SINR.
A. Receive Filter Design

In the first alternating step, we deal with the receive filter \( w \) design for a fixed \( R \). Specifically, we handle the optimization problem which is equivalently reformulated as

\[
\max_w \quad S \mathcal{N}^R(w) = \frac{w^H \Theta w}{w^H (\Phi + R_n) w},
\]

The optimization problem in (11) is a generalized Rayleigh quotient [17], which has an optimal solution

\[
w_{opt} = (\Phi + R_n)^{-1/2} \Phi [\Phi + R_n]^{-1/2} \Theta (\Phi + R_n)^{-1/2},
\]

where \( \Phi [\Phi + R_n]^{-1/2} \Theta (\Phi + R_n)^{-1/2} \) indicates the principle eigenvector associated with the largest eigenvalue of \( (\Phi + R_n)^{-1/2} \Theta (\Phi + R_n)^{-1/2} \).

B. Manifold-Based Waveform Optimization

In the next alternating step, given the fixed \( w \), the optimization problem (6) associated to the WCM can be equivalently recast as

\[
\max_R \quad S \mathcal{N}^R(R) = \frac{\text{tr} (\Gamma R)}{\text{tr} (\Sigma R)},
\]

s.t. \( \text{diag}(R) = \frac{P_0}{N_t L} \cdot 1 \), rank(\( R \)) = 1, \( R \succeq 0 \).

Notably, problem (13) is generally a NP-hard problem which can be approximatively solved by semidefinite relaxation (SDR). However, the SDR-based algorithm [11] is disadvantaged by the prohibitive computational burden. Therefore, we propose an efficient manifold-based method to solve problem (13).

The search region is a set of Hermitian, positive semidefinite (SDP), fixed-rank and fixed diagonal matrices, which suggests a quotient geometry for the complex fixed-rank ellipsoids. All \( R \) feasible for (13) have a unit diagonal, so that the search space of (13) is compact. This restriction is easily enforced by factoring \( R = Y Y^H \), yielding an equivalent problem

\[
\min_{Y \in \mathcal{M}} \quad f(Y) = -\frac{\text{tr} (\Gamma Y Y^H)}{\text{tr} (\Sigma Y Y^H)}, \tag{14}
\]

with the search space

\[
\mathcal{M} = \left\{ Y \in \mathbb{C}^{N_t L} : \text{diag}(Y Y^H) = \frac{P_0}{N_t L} \cdot 1 \right\}, \tag{15}
\]

being a smooth and compact manifold. It can be linearized at each point \( Y \) in \( \mathcal{M} \) by a tangent space

\[
\mathcal{T}_Y \mathcal{M} = \left\{ \xi_Y \in \mathbb{C}^{N_t L} : \text{diag}(\xi_Y Y^H + Y \xi_Y^H) = 0_{N_t L} \right\}. \tag{16}
\]

Endowing the tangent spaces of \( \mathcal{M} \) with the Euclidean metric \( g_Y(\xi, \eta) = \xi_Y^H \eta_Y \) turns \( \mathcal{M} \) into a Riemannian submanifold of Euclidean space \( \mathbb{C}^{N_t L} \). Herein, the smoothness of both the search space (15) and the cost function (14) make for straightforward second-order optimality conditions. Moreover, the retraction is a mapping from \( \mathcal{T}_Y \mathcal{M} \) to \( \mathcal{M} \). Herein, we define the retraction as

\[
\text{Retr}_Y(\xi_Y) = \frac{\xi_Y}{\|\xi_Y\|}. \tag{17}
\]

With the geometric structure of \( \mathcal{M} \) defined above, we can develop the Riemannian trust-region (RTR) algorithm to solve (14) in the following subsection.

C. Riemannian Trust-Region Algorithm

In order to minimize the cost function (14) on \( \mathcal{M} \), we need to compute the Riemannian gradient and the Riemannian Hessian of \( f(Y) \). The Riemannian gradient \( \nabla f(Y) \) satisfies

\[
g_Y(\xi_Y, \nabla f(Y)) = D_f(Y)[\xi_Y], \quad \forall \xi_Y \in \mathcal{T}_Y \mathcal{M}, \tag{18}
\]

where

\[
D_f(Y)[\xi_Y] = \lim_{t \to 0} \frac{f(Y + t \xi_Y) - f(Y)}{t}. \tag{19}
\]

Specifically, the Riemannian gradient \( \nabla f(Y) \) is the orthogonal projection of the Euclidean gradient of \( f(Y) \) to the tangent space \( \mathcal{T}_Y \mathcal{M} \), i.e.,

\[
\nabla f(Y) = \nabla f - Y \odot \text{Re} \{\nabla f \odot Y\}, \tag{20}
\]

where

\[
\nabla f = -\frac{2}{\rho^2} (\rho \Gamma Y - \mu \Sigma Y). \tag{21}
\]

is the Euclidean gradient of \( f(Y) \) with respect to \( Y \), with \( \rho = \text{tr}(\Sigma Y Y^H) \) and \( \mu = \text{tr}(\Gamma Y Y^H) \). The Riemannian Hessian of \( f(Y) \) at \( Y \) is a similarly restricted version of the Euclidean Hessian \( \text{ehess} f \) to the tangent space,

\[
\text{Hess}_f(Y)[\xi_Y] = \text{ehess} f - Y \odot \text{Re} \{\text{ehess} f \odot Y\}
- \xi_Y \odot \text{Re} \{\text{ehess} f \odot Y\}, \tag{22}
\]

with

\[
\text{ehess} f(Y) = -\frac{2}{\rho^2} \left[ \rho \Gamma \xi_Y + \text{tr}(\Sigma \xi_Y Y^H) \Gamma Y 
+ \text{tr}(\Sigma \xi_Y Y^H) \Gamma Y - \text{tr}(\Gamma \xi_Y Y^H) \Sigma Y 
- \text{tr}(\Gamma \xi_Y Y^H) \Sigma Y - \text{tr}(\Gamma Y Y^H) \Sigma \xi_Y 
+ \frac{4}{\rho^3} \left\{ \rho \Gamma Y - \text{tr}(\Gamma Y Y^H) \Sigma Y \right. 
\left. \times \left[ \text{tr}(\Sigma \xi_Y Y^H) + \text{tr}(\Sigma \xi_Y Y^H) \right] \right\}. \tag{23}
\]

With the Riemannian gradient and the Riemannian Hessian at hand, we develop the Riemannian trust-region (RTR) algorithm to our problem. The trust-region algorithm solves the problem

\[
\min_{\xi_Y \in \mathcal{T}_Y \mathcal{M}} \quad f(Y) + g_Y(\nabla f(Y), \xi_Y)
+ \frac{1}{2} g_Y(\text{Hess}_f(Y)[\xi_Y], \xi_Y), \tag{24}
\]

s.t. \( g_Y(\xi_Y, \xi_Y) \leq \delta^2 \), where \( \delta \) is the trust-region radius.

The computational cost of the RTR algorithm is significantly less compared with the SDR-based method [11] mainly due to the following reason. In the RTR method, the randomization procedure can be avoided, since the optimized \( Y \) is actually the optimum waveform vector \( \mathbf{s}_{opt} \). Moreover, the RTR method can be significantly efficient when the explicit Riemannian gradient and Riemannian Hessian are provided.
TABLE I
JOINT OPTIMIZATION OF TRANSMIT WAVEFORM AND RECEIVE FILTER VIA MAO ALGORITHM FOR MIMO-STAP RADARS

<table>
<thead>
<tr>
<th>Input: $R^0$</th>
<th>Output: $s_{opt} = Y^*<em>{s</em>{opt}}$, $w_{opt}$</th>
</tr>
</thead>
</table>
| For $i = 1, 2, \ldots, \text{do}$ | \begin{enumerate} 
  1. $R^i = R^0$;
  2. Optimize $w^i$ by (12);
  3. Optimize $R^i = Y Y^H$ via RTR algorithm;
  4. $i \leftarrow i + 1$;
  5. until $\frac{STN{R}^{(i+1)} - STN{R}^{(i)}}{STN{R}^{(i)}} \leq \varepsilon$ for some $\varepsilon > 0$, where $STN{R}^{(i)}$ is the output SINR at the $i$-th iteration. \end{enumerate} |

Fig. 2. Output SINR against the number of iterations. The total computing time of the MAO algorithm and the SDR-based algorithm are, respectively, 46.97 s and 314.29 s.

The procedures of the joint optimization of transmit waveform and receive filter via the MAO algorithm are summarized in Table I.

IV. SIMULATION RESULTS

In this section, we provide several numerical examples to demonstrate the performance of the MAO algorithm. We used the MATLAB implementation of the RTR method [18] in our simulations. Consider an airborne MIMO-STAP radar with $N_t = 4$, $N_r = 4$, and $d_t = 2\lambda$ and $d_r = \lambda/2$. The carrier frequency and bandwidth of the waveform are $f_0 = 1$ GHz and $B = 1$ MHz. The altitude of the radar platform is 9 km and the platform is moving with a speed $V_P = 200$ m/s. The number of pulses in a CPI is $K = 16$, and the code length is $L = 16$. PRF is set to $f_r = 4V_P/\lambda$ and $B = 1$ MHz. The target is at range $R = 13$ km with $\phi_T = \arcsin (H/R)$ and $\theta_T = 0^\circ$. The normalized Doppler frequency of the target is $f_D = 0.3$. The number of discrete azimuth sectors of the interested rang ring is $N_c = 361$ and $P = 1$. We assume that $\sigma_{T,p,q} = 1$. The total transmit power is $P_0 = 100$. The terminating criteria is $\varepsilon = 10^{-3}$. For the RTR method, the upper bound for the trust-region radius $\delta$ is chosen as [12]

$$\delta = \sqrt{N_r L + K N_t L}. \quad (25)$$

The initial trust-region radius is chosen as $\delta/8$.

First, we analyze the convergence of the MAO algorithm compared with the Algorithm 2 in [11], which is an SDR-based alternating algorithm. Fig. 2 shows the output SINRs of the two algorithms versus the number of iterations. Under the same terminating criteria, the MAO algorithm converges at the 22-th iteration, while the SDR-based algorithm converges at the 35-th iteration. Moreover, the total computing time of the MAO algorithm and the SDR-based algorithm are, respectively, 46.97 s and 314.29 s. It is evident that the MAO algorithm converges faster than the SDR-based algorithm. In addition, the SINRs associated with the MAO algorithm and the SDR-based algorithm are, respectively, 40.63 dB and 38.79 dB. Thus, the MAO algorithm is computationally more efficient and achieves higher SINR value than the SDR-based algorithm.

In Fig. 3, we present the spatial-temporal beampatterns associated with the MAO algorithm, the SDR-based algorithm, the SIMO-STAP, and the orthogonal waveform. The corresponding receive filters of the coherent waveform and the orthogonal waveform are calculated via (14). We can observe from Fig. 3 that all four beampatterns have deep nulls in the clutter ridge and significant high gain in the target direction. The response in the target direction of the MAO algorithm is the strongest, followed by the SDR-based algorithm, the SIMO-STAP and the orthogonal waveform. The superiority of the MAO algorithm is once again highlighted in Fig. 3.

V. CONCLUSION

In this letter, we addressed the joint design for a colocated MIMO radar to maximize the SINR in the presence of signal-dependent interference. We proposed a manifold-based alternating optimization (MAO) approach, which views the considered problem as an unconstrained optimization on a manifold. We presented the geometrical structure of the feasible sets in the form of a Riemannian manifold and derived the explicit Riemannian gradient and Hessian operators. Thus, the Riemannian trust-region (RTR) method can be developed to solve the problem. The monotonically increasing SINR at each iteration and the convergence of the MAO algorithm were guaranteed. In addition, our proposed algorithm reduced the computational cost compared with the conventional SDR-based algorithm that operates in Euclidean space. Furthermore, the manifold-based optimization framework can be extended to tackle more challenging constraints on the joint design.
REFERENCES


