Circular Acoustic Vector-Sensor Array for Mode Beamforming

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ABSTRACT—Undersea warfare relies heavily on acoustic means to detect a submerged vessel. The frequency of the acoustic signal radiated by the vessel is typically very low, thus requires a large array aperture to achieve acceptable angular resolution. In this paper, we present a novel approach for low-frequency direction-of-arrival (DOA) estimation using miniature circular vector-sensor array mounted on the perimeter of a cylinder. Under this approach, we conduct beamforming using decomposition in the acoustic mode domain rather than frequency domain, to avoid the long wavelength constraints. We first introduce a multi-layer acoustic gradient scattering model to provide a guideline and performance predication tool for the mode beamformer design and algorithm. We optimize the array gain and frequency response with this model. We further develop the adaptive DOA estimation algorithm based on this model. We formulate the Capon spectra of the mode beamformer which is independent of the frequency band after the mode decomposition. Numerical simulations are conducted to quantify the performance and evaluate the theoretical results developed in this study.

INDEX TERMS—Acoustic signal processing, acoustic transducers, array signal processing, direction-of-arrival estimation.

I. INTRODUCTION

TRADITIONALLY, the passive approach for underwater acoustic surveillance consists of installing a long linear array in the ocean, with omnidirectional hydrophones as the sensing elements. For an array of pressure sensors, the array gain in directivity is fundamentally limited by the size of the array. In littoral or shallow water environment, especially areas with heavy shipping density, the complex ambient noise characteristics and varying ocean bottom conditions often impair the ability to detect the acoustic signal produced by the underwater target with the long array [1], [2].

The radiated noise produced by the vessel’s machinery and its motion in sea consists of broadband and narrowband components. The propeller and the hydrodynamic turbulence produce a broadband component whereas the propeller, propulsion system, and auxiliary machinery generate the narrowband component. Generally, the frequency of both narrowband and broadband components fall within a few tens to a few hundreds of Hertz [3], [4], [37]. Such a low-frequency band requires a large array aperture to achieve the desired angular resolution.

Cylindrical hydrophone arrays have been studied over the past three decades and several practical systems have been built and demonstrated [6], [7]. The circular arrays can avoid the left-right ambiguity on bearing estimation that linear arrays normally encounter. However, the diameter of an underwater surveillance circular array usually reaches a few meters as it is normally used to sense the low frequency components of the acoustic sources. The huge array frame makes its deployment, maintenance, and recovery difficult.

A novel acoustic vector sensor, which measures the acoustic pressure and acoustic particle velocity field simultaneously, has been proven to be very effective for shallow water surveillance [1]. The acoustic vector sensors have been used in directional sonobuoy for the past half-century [35]. Nehorai and Paldi [22] first developed the measurement model of the acoustic vector sensor (AVS) array and analyzed the theoretical performance bounds on the direction finding. The DOA estimation using AVS array was further investigated by many researchers [23]–[25]. New application areas such as near field holography [26] and air acoustical signal extraction [27] have been explored by both academic and industrial communities.

A conventional acoustic vector sensor consists of an omnidirectional pressure sensor and a particle velocity sensor. It simultaneously measures the acoustic pressure and three orthogonal components of the particle velocity at single point in an acoustic field by single snapshot [28], [29]. Cray proved a single vector sensor can provide 6 dB of directivity gain [30]. One of our goals is to propose a miniaturized cylindric vector array that can provide much better DOA estimation ability.

In this paper, we introduce a novel acoustic vector-sensor array architecture and its associated DOA estimation algorithm. In Section II we discuss the physical acoustic model of our approach. We introduce the acoustic signal model of a circular vector-sensor array which senses both the radial and tangential components of the acoustic field gradient. The miniature array is mounted in the perimeter of an acoustically coated cylinder. We explore the mode domain decomposition of the acoustic field that is the basis for the proposed DOA estimation algorithm. In Section III we develop the adaptive DOA estimation algorithm. We show that for the novel algorithm, the array manifold is no longer a function of wavelength. Instead, the array DOA estimation performance is mainly decided by the acoustic and physical characteristics of the cylinders. We illustrate and numerically prove that by coating the cylinder, the required mode number and frequency response curve can be created to fit the incident acoustic waves. This enables our approach to pro-
vide low-frequency broadband beamforming with miniaturized array aperture comparing with conventional methods. Numerical simulations are presented in Section IV. The impact of the modal coefficients on the beamforming bandwidth is numerically assessed. The mode beamformer array gain and beam pattern are computed and a design example based on practical material acoustic characteristics are illustrated. Section V concludes our findings and results of this paper.

II. PHYSICAL MODEL OF THE CYLINDRICAL MODE ARRAY

We consider a harmonic, plane acoustic wave impinging on a coated cylinder shown in Fig. 1. The cylinder is positioned vertically at the origin of the cylindrical coordinate system. The axis of the cylinder is taken as the \( z \)-axis. This cylindrical sensor node is submerged in water. Denote the water density and sound velocity in the water by \( \rho \) and \( c \), respectively. The outer layer of radius \( R_c \) is made of viscoelastic material with density of \( \rho_2 \). Its elastic and viscous Lamé constants \( [15] \) are \( \lambda_{e2} \), \( \mu_{e2} \), \( \lambda_{v2} \), and \( \mu_{v2} \), respectively. The cylindrical array frame is made of elastic material of radius \( R_b \) with elastic Lamé constants of \( \lambda_{e3} \) and \( \mu_{e3} \), respectively. The inner cylinder compartment of radius \( R_a \) is filled with air of density \( \rho_1 \) and sound velocity \( c_4 \).

Assume a unit-magnitude steady-state plane acoustic wave with radial frequency \( \omega \) traveling in the direction of azimuth angle \( \phi_b \). Without loss of generality, we assume the incident elevation angle is \( \pi/2 \). This assumption is normally true when the underwater source is far enough comparing with the water depth. The pressure of the incident wave can be written as \([9],[18],[19] \)

\[
 p_i = e^{i(\omega t + \mathbf{kr})} = \left\{ \sum_{n=0}^{\infty} j^n \epsilon_n B_n(kr) \cos(n(\phi - \phi_b)) \right\} e^{i\omega t} \tag{1}
\]

where \( j^2 = -1 \) and \( \phi \) is the scattering angle

\[
\mathbf{t} = r(\cos \phi_b, \sin \phi) T
\]

where \( r \) is the distance from the observation to the origin point of the coordinate system

\[
\mathbf{k} = k(\cos \phi_b, \sin \phi) T
\]

where \( k = \omega/c \) is the wave number in water, \( J_n(\cdot) \) is the \( n \)-th order Bessel function of the first kind, and \( \epsilon_n \) is the Neumann function

\[
\epsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1, 2, 3, \ldots \end{cases}
\]

The scattered acoustic wave can be written as \([10] - [17] \)

\[
 p_s = \left\{ \sum_{n=0}^{\infty} j^n \epsilon_n B_n(kr) \cos(n(\phi - \phi_b)) \right\} e^{i\omega t} \tag{2}
\]

where \( H_n^{(1)}(\cdot) \) is the Hankel function of the first kind. The reflection coefficient \( B_n(\cdot) \) is determined by the boundary conditions that the pressure and radial component of the particle velocity must be continuous at the boundaries \([20],[21] \).

Governed by the Helmholtz equations, the scalar acoustic potentials in the viscoelastic coating and the node frame can be represented as

\[
\Theta_2 = \left\{ \sum_{n=0}^{\infty} j^n \epsilon_n \left[ B_n(2) J_n(k_{d2}r) + B_n(3) Y_n(k_{d2}) \right] \right\} e^{i\omega t},
\]

\[
\Psi_2 = \left\{ \sum_{n=0}^{\infty} j^n \epsilon_n \left[ B_n(4) J_n(k_{d2}r) + B_n(5) Y_n(k_{d2}) \right] \right\} e^{i\omega t},
\]

\[
\Theta_3 = \left\{ \sum_{n=0}^{\infty} j^n \epsilon_n \left[ B_n(6) J_n(k_{d3}r) + B_n(7) Y_n(k_{d3}) \right] \right\} e^{i\omega t},
\]

\[
\Psi_3 = \left\{ \sum_{n=0}^{\infty} j^n \epsilon_n \left[ B_n(8) J_n(k_{d3}r) + B_n(9) Y_n(k_{d3}) \right] \right\} e^{i\omega t} \tag{3}
\]

where the \( Y_n(\cdot) \) is the \( n \)-th order Neumann function. For the viscoelastic outer layer material

\[
k_{d2} = \frac{\omega}{c_{d2} \sqrt{1 - j\omega M_2}},
\]

\[
k_{d3} = \frac{\omega}{c_{d3}},
\]

\[
k_{s2} = \frac{\omega}{c_{s2} \sqrt{1 - j\omega N_2}},
\]

\[
k_{s3} = \frac{\omega}{c_{s3}} \tag{4}
\]

where

\[
c_{s2} = \sqrt{\frac{\mu_{e2}}{\rho_2}},
\]

\[
c_{s3} = \sqrt{\frac{\mu_{e3}}{\rho_3}},
\]

\[
c_{d2} = \sqrt{\frac{\lambda_{e2} + 2\mu_{e2}}{\rho_2}},
\]

\[
c_{d3} = \sqrt{\frac{\lambda_{e3} + 2\mu_{e3}}{\rho_3}}.
\]

\[
M_2 = \frac{\lambda_{e2} + 2\mu_{e2}}{\lambda_{e2}},
\]

\[
N_2 = \frac{\mu_{e2}}{\mu_{e2}}.
\]
The acoustic field in the cylinder is

\[ p_4 = \left\{ \sum_{n=0}^{\infty} j^n \varepsilon_n B_n(10) J_n(k_4 r) \cos(n(\phi - \phi_b)) \right\} e^{j\omega t} \tag{9} \]

where

\[ k_4 = \frac{\omega}{c_4}. \tag{10} \]

The coefficients of (5), (6), and (9) can be solved as

\[ Db_B = a_A \tag{11} \]

where [see (12) and (13) shown at bottom of page] and

\[ b_B = [B_n(1), B_n(2), B_n(3), \ldots, B_n(10)]^T_{1 \times 10}. \tag{14} \]

The details of the expressions of the elements of (12) to (13) are given in the Appendix.

The total complex acoustic pressure in the perimeter of the cylinder is

\[ p_T = (p_i + p_a). \tag{15} \]

Denote

\[ B_{n,\phi} = B_n(1) \tag{16} \]

then combining (1) and (5) yields

\[ p_T = \left\{ \sum_{n=0}^{\infty} C_n \cos n(\phi - \phi_b) \right\} e^{j\omega t} \tag{17} \]

where

\[ C_n = j^n \varepsilon_n \left\{ J_n'(kr) + B_{n,\phi} H_n^{(1)}(kr) \right\}. \tag{18} \]

Equations (19) and (20) can be further denoted as

\[ v_T = \left\{ \sum_{n=0}^{\infty} D_n \cos[n(\phi - \phi_b)] \right\} e^{j\omega t} \tag{19} \]

\[ v_\phi = \left\{ \sum_{n=1}^{\infty} E_n \sin[n(\phi - \phi_b)] \right\} e^{j\omega t} \tag{20} \]

It can be observed from (21) that both the radial and tangential gradient of the acoustic field can be decomposed into a series of cosine and sine components. We define \( D_n \cos[n(\phi - \phi_b)] \) as the \( n^{th} \) mode component of the radial gradient with mode coefficient \( D_n \) and \( E_n \sin[n(\phi - \phi_b)] \) as the \( n^{th} \) mode component of the tangential gradient with mode coefficient \( E_n \), respectively.

## III. CIRCULAR VECTOR-SENSOR ARRAY DOA ESTIMATION

In this section, we establish the rationale for DOA estimation based upon the physical model we introduced in Section II. The outputs of the circular vector-sensor array are transformed into mode domain subspace and spatially filtered to produce the DOA estimation.

\[
D = \begin{bmatrix}
    d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & 0 & 0 & 0 & 0 & 0 \\
    d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & 0 & 0 & 0 & 0 & 0 \\
    0 & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & 0 & 0 & 0 & 0 & 0 \\
    0 & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} & d_{4,7} & d_{4,8} & d_{4,9} & 0 \\
    0 & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} & d_{5,7} & d_{5,8} & d_{5,9} & 0 \\
    0 & d_{6,2} & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6} & d_{6,7} & d_{6,8} & d_{6,9} & 0 \\
    0 & d_{7,2} & d_{7,3} & d_{7,4} & d_{7,5} & d_{7,6} & d_{7,7} & d_{7,8} & d_{7,9} & 0 \\
    0 & 0 & 0 & 0 & 0 & d_{8,6} & d_{8,7} & d_{8,8} & d_{8,9} & d_{8,10} \\
    0 & 0 & 0 & 0 & 0 & d_{9,6} & d_{9,7} & d_{9,8} & d_{9,9} & d_{9,10} \\
    0 & 0 & 0 & 0 & 0 & d_{10,6} & d_{10,7} & d_{10,8} & d_{10,9} & d_{10,10}
\end{bmatrix}_{10 \times 10} 
\tag{12} \]

\[
a_A = [a_1, a_2, 0, \ldots, 0]^T_{1 \times 10}. \tag{13} \]
Integrating the radial component \( v_r \) of (21) over \( \phi \), we have

\[
\int_0^{2\pi} v_r \cos m\phi d\phi = \sum_{n=0}^{\infty} \left( \cos n\phi \right) \cos m\phi d\phi
\]

\[
= \sum_{n=0}^{\infty} D_n \cos \left[ n(n - \phi_s) \right] \cos m\phi d\phi
\]

\[
= \sum_{n=0}^{\infty} D_n \left\{ \cos n\phi_s \int_0^{2\pi} \cos n\phi \cos m\phi d\phi + \sin n\phi_s \int_0^{2\pi} \sin n\phi \cos m\phi d\phi \right\}
\]

(23)

Observe that we dropped the time dependence of (21) for easier readability. Using the following integral identities [31], [32]:

\[
\int_0^{2\pi} \cos n\phi \cos m\phi d\phi = \delta_{nm}
\]

\[
\int_0^{2\pi} \sin n\phi \cos m\phi d\phi = 0
\]

(24)

where \( \delta_{nm} \) is the Kronecker function defined as

\[
\delta_{nm} = \begin{cases} 
1 & n = m \\
0 & n \neq m.
\end{cases}
\]

Taking (24) and (25) into (23) yields

\[
\cos m\phi_s = \frac{1}{\pi D_m} \int_0^{2\pi} v_r(\phi) \cos m\phi d\phi
\]

(26)

where \( \phi_s \) is the acoustic source incident angle, as defined in Section II. Since the acoustic sensors can sense the acoustic gradient only discretely, by assuming the \( N \) sensors are placed equally along the perimeter of the cylinder and \( N \) satisfies the Shannon’s sample theorem, namely \( N > 2M_{\text{max}} + 1 \) where \( M_{\text{max}} \) is the highest order of the mode; the discrete form of (26) becomes

\[
\cos m\phi_n = \frac{2}{N D_m} \sum_{n=0}^{N-1} v_r(\phi_n) \cos m\phi_n
\]

(27)

where

\[
\phi_n = \frac{n 2\pi}{N}, \quad n = 0, 1, 2, \ldots, N - 1.
\]

(28)

Similarly, by transforming the tangential component of (21) over \( \phi \), we have

\[
\sin m\phi_n = \frac{-2}{N} E_m \sum_{n=0}^{N-1} v_r(\phi_n) \cos m\phi_n.
\]

(29)

We consider a two-dimensional gradient vector-sensor array of \( N \) elements uniformly placed in the perimeter of the cylinder. Each vector sensor measures both the radial and tangential components of the incident acoustic field. The received radial and tangential components at snapshot \( t_k \) are

\[
v_r(t_k) = \left[ v_r(\phi_s) e^{j\omega t_k}, \ldots, v_r(\frac{2(N-1)\pi}{N} - \phi_s) e^{j\omega t_k} \right]^T
\]

(30)

and

\[
v_\phi(t_k) = \left[ v_\phi(\phi_s) e^{j\omega t_k}, \ldots, v_\phi(\frac{2(N-1)\pi}{N} - \phi_s) e^{j\omega t_k} \right]^T.
\]

(31)
By transforming the radial and tangential particle velocity components following (27) and (29) and taking the linear sum of the output of the transformation at time $t_k$, the decomposed mode coefficients of the radial are

$$g_r(t_k) = W_r \phi(t_k)$$

(32)

where

$$g_r(t_k) = [g_{r,0}(t_k), \ldots, g_{r,M-1}(t_k)]^T_{(M \times 1)}$$

(33)

[see (34) at the bottom of the page] and for the tangential gradients are

$$h_\phi(t_k) = W_\phi \phi(t_k)$$

(35)

where

$$h_\phi(t_k) = [h_{\phi,1}(t_k), \ldots, h_{\phi,M-1}(t_k)]^T_{(M-1) \times 1}$$

(36)

[see (37) shown at the bottom of the page].

Define a new acoustic decomposition vector

$$u(t_k) = \left[ \begin{array}{c} g_r(t_k) \\ h_\phi(t_k) \end{array} \right]_{(2M-1) \times 1}$$

(38)

By applying a spatial filter to the transformed subspace vector $u$, the output of the filter and sum beamformer yields

$$s(t_k) = a(\phi)^H u(t_k)$$

(39)

where

$$a(\phi) = [1, \ldots, \cos((M-1)\phi), \sin\phi, \ldots, \sin((M-1)\phi)]^T_{(2M-1) \times 1}$$

(40)

Given $K$ temporal samples $s(t_0), s(t_1), \ldots, s(t_{K-1})$, the Capon spectra is [5]

$$P(\phi) = \frac{1}{a(\phi)^H R^{-1} a(\phi)}$$

(41)

where

$$R = \frac{1}{K} \sum_{k=0}^{K-1} u(t_k) u(t_k)^H$$

(42)

is the covariance matrix of the acoustic gradient field decomposition output.

IV. NUMERICAL EXAMPLES

In this section, we illustrate by numerical examples some of the points made in Sections II and III.

A. Modal Coefficients

Fig. 3 shows plots of the radial and tangential gradient acoustic field for $kR_c = 1, 5, $ and 10, respectively, following (21) and (22).

Without loss of generality, we adopt the following parameters for our simulations shown in Table I [15]. We assume that the far-field acoustic wave impinges on the sonar node with azimuth angle of $\pi$. Observation of Fig. 3 shows a few facts. The radial acoustic gradient varies more significantly with the variation of $kR_c$ than with the tangential component. When $kR_c$ is small, the radial gradient signal energy is distributed on the back scattering region. When $kR_c$ increases, strong signal occurs in the forward scattering region. The tangential component is present at both the forward and back scattering regions. With the increase of $kR_c$, the tangential gradient energy is narrowed more to the backscattering area.

Fig. 4 shows the mode coefficients of the decomposed radial and tangential gradient of the acoustic field following (11)–(22).
Fig. 3. Magnitude of radial (left) and tangential (right) acoustic gradient field of the scattering signal with $kR_c = 1$ (top), $kR_c = 5$ (middle) and $kR_c = 10$ (bottom).

The simulation parameters are the same as shown in Table I. Observation of the plot shows that up to 6 modes of the radial gradient component can be activated above 0 dB when $kR_c$ is greater than 2. The modes are usable for broadband beamforming until $kR_c$ reaches about 5. For the tangential component, Mode 0 is not able to be activated and the magnitude of the mode coefficients in most situations is weaker than radial component. The mode 1 decays the fastest for both the radial and tangential components with the increase of $kR_c$. However, the first notch of mode 1 of the radial component has the strength of about 12 dB at $kR_c$ about 3.8 while the counter part of the tangential component drops sharply to below −12 dB. So for high signal-to-noise ratio (SNR) condition, adopting both radial and tangential components is more appropriate to increase the array gain whereas for low SNR, utilizing the radial compo-

### TABLE I

<table>
<thead>
<tr>
<th>SIMULATION PARAMETERS</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water density</td>
<td>$10^3$ kg/m$^3$</td>
</tr>
<tr>
<td>Sound speed in water</td>
<td>1500 m/sec</td>
</tr>
<tr>
<td>Outer layer density</td>
<td>$1.13 \times 10^3$ kg/m$^3$</td>
</tr>
<tr>
<td>Outer layer Lamé constant $\lambda_c$</td>
<td>$2.218 \times 10^6$ N/m$^2$</td>
</tr>
<tr>
<td>Outer layer Lamé constant $\mu_c$</td>
<td>$1 \times 10^7$ N/m$^2$</td>
</tr>
<tr>
<td>Outer layer viscous Lamé constant $\lambda_v$</td>
<td>$1.5824 \times 10^6$ N/m$^2$</td>
</tr>
<tr>
<td>Outer layer viscous Lamé constant $\mu_v$</td>
<td>$1.5824 \times 10^6$ N/m$^2$</td>
</tr>
<tr>
<td>Outer layer radius</td>
<td>0.2667 m</td>
</tr>
<tr>
<td>Inner layer density</td>
<td>$7.84 \times 10^3$ kg/m$^3$</td>
</tr>
<tr>
<td>Inner layer Lamé constant $\lambda_c$</td>
<td>$1.13 \times 10^7$ N/m$^2$</td>
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<tr>
<td>Inner layer Lamé constant $\mu_c$</td>
<td>$7.54 \times 10^7$ N/m$^2$</td>
</tr>
<tr>
<td>Inner layer radius</td>
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</tr>
<tr>
<td>Air density</td>
<td>$1.2 \times 10^{-5}$ kg/m$^3$</td>
</tr>
<tr>
<td>Sound speed in air</td>
<td>344 m/sec</td>
</tr>
<tr>
<td>Equipment canister radius</td>
<td>0.2541 m</td>
</tr>
</tbody>
</table>
ponent is more appropriate to ensure the detection capability for the beamforming.

The flat frequency response area can be controlled by the node physical dimension and material characteristics. Fig. 5 shows the curve of the mode 1 frequency response as a function of the outer layer thickness. Observations of Fig. 5 show that when \( kR_c < 1.4 \), the mode 1 coefficient increases with the increase of the outer layer thickness. However, when \( kR_c > 1.4 \), the mode 1 coefficient decreases with the increase of the thickness. By controlling the physical dimension of the frame and material characteristics, the usable frequency bandwidth can be controlled.

### B. Beam Pattern and Array Gain

Fig. 6 shows the single mode beam pattern of the decomposed radial and tangential acoustic gradient field following (26) and (29). After the field decomposition, the decomposed mode is no longer a function of frequency, namely the beam pattern is frequency independent. It forms a very unique feature for applications requiring frequency invariant constant beam steering. Among them are speech capture, battle field acoustic information collection etc. Theoretically an infinite mode number can be excited to increase the directionality of the beamformer. However, in practical implementations, higher mode might be buried in the ambient noise due to its weak amplitude. The array designer has to assess the array frame architecture to select the appropriate frequency band and mode number, as we discussed in Section IV-A.

The WNG (white noise gain) of an array is defined as

\[
\text{WNG} = \frac{|\mathbf{a}^H \mathbf{d}_m|^2}{\mathbf{a}^H \mathbf{a}}
\]  

where \( \mathbf{a} \) denotes the coefficient of the beamformer and \( \mathbf{d} \) is the representation which depends on the array geometry and source direction [36].

We compare the mode decomposition vector sensor array beamformer with conventional and vector sensor circular array beamformers, of the same size and number of sensors. The WNG of our mode decomposition vector-sensor array is

\[
\text{WNG}_m = \frac{|\mathbf{a}^H_m \mathbf{d}_m|^2}{\mathbf{a}^H_m \mathbf{a}_m}
\]  

where \( \mathbf{a}_m \) is defined by (40) and

\[
\mathbf{d}_m = [1, \ldots, \cos((M-1)/\phi_m), \sin(\phi_m), \ldots, \sin((M-1)/\phi_m)]^T_{(2 \times M-1) \times 1}
\]  

Similarly, the conventional circular array WNG can be denoted

\[
\text{WNG}_c = \frac{|\mathbf{a}^H \mathbf{a}_c|^2}{\mathbf{a}^H \mathbf{a}_c}
\]  

where

\[
\mathbf{a}_c(\phi) = [a_{c0}(\phi), \ldots, a_{cN-1}(\phi)]^T
\]

\[
a_{cN}(\phi) = e^{-i k R \cos(2\pi(n-1)/N)} \cos(\phi) + \sin(2\pi(n-1)/N) \sin(\phi)
\]
and
\[ d_s = [d_{s,0}(\phi_s), \ldots, d_{s,N-1}(\phi_s)]^T \quad (49) \]
where
\[ d_{s,\nu}(\phi_s) = e^{-ikR(\cos(2\pi(n-1)/N) \cos(\phi_s) + \sin(2\pi(n-1)/N) \sin(\phi_s))} \cdot \sum_{n=0}^{N-1} \frac{e^{ikR(\cos(2\pi(n-1)/N) \cos(\phi_s) + \sin(2\pi(n-1)/N) \sin(\phi_s))}}{a_{\nu,n}^R} \quad (50) \]

The conventional circular vector-sensor array WNG is
\[ WNG_v = \frac{\left| \sum_{\nu=0}^{N-1} d_{s,\nu} \right|^2}{\sum_{\nu=0}^{N-1} a_{\nu}^R a_{\nu}^*} \quad (51) \]
where
\[ a_{\nu}(\phi) = [a_{\nu,0}(\phi), \ldots, a_{\nu,N-1}(\phi)]^T \quad (52) \]
\[ a_{\nu,n}(\phi) = \frac{-ik [\cos(2\pi(n-1)/N) \cos(\phi) + \sin(2\pi(n-1)/N) \sin(\phi)]}{a_{\nu,n}^R} \quad (53) \]
\[ a_{\nu,n}(\phi) = \frac{-ik [\cos(2\pi(n-1)/N) \sin(\phi)]}{a_{\nu,n}^R} \quad (54) \]
and
\[ d_v(\phi_s) = [d_v,0(\phi_s), \ldots, d_v,N-1(\phi_s), \ldots, d_v,0(\phi_s), \ldots, d_v,N-1(\phi_s)]^T \quad (55) \]
where
\[ d_{v,\nu}(\phi_s) = \frac{-ik [\cos(2\pi(n-1)/N) \cos(\phi_s) + \sin(2\pi(n-1)/N) \sin(\phi_s)]}{a_{\nu,n}^R} \quad (56) \]
\[ d_{v,\nu}(\phi_s) = \frac{-ik [\cos(2\pi(n-1)/N) \sin(\phi_s)]}{a_{\nu,n}^R} \quad (57) \]

where \( R \equiv R_c \) and the number of sensors is \( 2M \). \( M \) is the maximum mode number for the proposed beamformer.

The white noise array gain of the conventional, vector sensor and the proposed circular array are shown in Figs. 7, 8, and 9, respectively. It can be clearly observed that under the same array physical dimension, our approach outperforms the conventional array significantly on beam width and sidelobe perspective.

C. Design Example

We next design a sonar node using the physical and acoustic specifications listed in Table I. We assume an acoustic source impinging from far field at an incident angle of 60°. The synthetic signal is generated following (19) and (20). The maximum mode number of 16 is chosen to produce the synthetic signal.
To fulfill the spatial Nyquist sample role, 12 vector sensors are evenly distributed in the perimeter of the cylinder each measuring both the radial and tangential gradient component of the incident acoustic field. The signal is selected to be narrow band at 600 Hz. The signal is further digitalized at sample rate of 8 kHz.

The maximum mode of the beamformer is selected as 5. Every 2048 points of data is used to estimate the covariance matrix of the beamformer. The sensor signal is converted to mode domain using (27) and (28). Then the 2048 points transformed data are used for DOA estimation.

For comparison, we also calculate the beamforming results based upon the conventional beamforming approaches. Under the conventional far field assumption, the unit-magnitude field measured at sensor \( n \) and due to a source at azimuthal DOA \( \phi \) is given by \([5]\)

\[
x = e^{r^T k}
\]

where

\[
k = k(\cos(\phi), \sin(\phi))^T
\]

and

\[
r = R \left( \cos\left(\frac{2\pi(n-1)}{N}\right), \sin\left(\frac{2\pi(n-1)}{N}\right) \right)^T.
\]

So the scalar, radial and tangential components are

\[
v_n = e^{-jk \cos\left(\frac{2\pi(n-1)}{N}\right) \sin(\phi)}
\]

\[
v_r,n = k \left( \cos\left(\frac{2\pi(n-1)}{N}\right) \cos(\phi) + \sin\left(\frac{2\pi(n-1)}{N}\right) \sin(\phi) \right)
\]

\[
v_\phi,n = k R \left( \cos\left(\frac{2\pi(n-1)}{N}\right) \sin(\phi)
\]

\[
+ \sin\left(\frac{2\pi(n-1)}{N}\right) \cos(\phi) \right)
\]

(61)

where \( N \) is the sensor number. The measurement of the scalar array output is

\[
s(t_k) = a_n(\phi)^H u_n(t_k)
\]

and

\[
u_n(t_k) = [v_0(t_k), \ldots, v_n(t_k), \ldots, v_{N-1}(t_k)]^T.
\]

The conventional circular vector array output measurement is

\[
s_c(t_k) = a_n(\phi)^H u_c(t_k)
\]

and

\[
u_c(t_k) = [v_{r,0}(t_k), \ldots, v_{r,N-1}(t_k), v_{\phi,0}(t_k), \ldots, v_{\phi,N-1}(t_k)]^T.
\]

The computation results are shown in Fig. 10. It is obvious that under the same simulation conditions, although all three beams peak at 60°, the angle the beam drops 3 dB occurs at 5° or mode beamforming whereas the counterpart of conventional

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**Fig. 7.** The white noise array gain of a conventional circular array with \( N = 11 \). The bearing of the source is \( \pi/6 \), the array radius is .26 m, and the sound speed in water is 1500 m/s.

**Fig. 8.** White noise array gain of the conventional circular vector sensor array with \( N = 11 \). The bearing of the source is \( \pi/6 \), the array radius is .26 m, and the sound speed in water is 1500 m/s.

**Fig. 9.** White noise array gain of the proposed mode decomposition vector sensor array with \( M = 5 \). The bearing of the source is \( \pi/6 \), the array radius is .26 m, and the sound speed in water is 1500 m/s.
We conclude that there are two advantages of our algorithm over the conventional circular array or a single vector-sensor array. Firstly, our approach processes the signals in mode domain, the DOA performance and the array gain rely mostly on the highest mode that can be activated. The model we developed was used to show how the highest mode and frequency response curve can be estimated and are controlled by the coating thickness. We have found by numerical simulation that a 26-cm array aperture can accurately estimate a narrow band acoustic source at 600 Hz.

The second advantage our algorithm produces is that once the signal is converted from the sensor domain to the mode domain, the beamformer performance is actually independent of the signal frequency. For practice, we can first separate the signal into narrow band frequencies and conduct mode decomposition using precalculated mode coefficients $D_m$ and $E_m$. After that, the signal at each beam is beamformed at the mode domain with the same mode number hence same beamwidth. This forms an important feature for certain applications such as speech capture or acoustic intelligent information collection. Although our current study only covers the narrow band DOA scenario, we will aim to show more benefits of our approach for broadband beamforming in our future work.

We confined our study to azimuth domain DOA estimation but the model can be easily extended to both azimuth and elevation DOA estimation. The current research will readily extend to two-dimensional DOA estimation. Our study is mainly focused on underwater acoustic sensing. However since the electromagnetic (EM) scattering mechanism is quite similar to acoustic scattering, the algorithm is likely to be extended to be implemented to electromagnetic wave DOA estimation.

APPENDIX

COEFFICIENTS OF THE SCATTERING MATRIX

The elements of matrix $D$ in (12) are given as follows:

$$d_{1,1} = \frac{\rho}{\rho^2} k_{s_2} k_{r_c}^2 \frac{H_1^{(1)}(k_{s_2} R_c)}{J_n(k_{s_2} R_c)}$$
$$d_{1,2} = 2(n^2 - k_{s_2}^2) Y_n(k_{s_2} R_c) - 2k_{s_2} R_c Y_n'(k_{s_2} R_c)$$
$$d_{1,3} = 2(n^2 - k_{s_2}^2) Y_n(k_{s_2} R_c) - 2k_{s_2} R_c Y_n'(k_{s_2} R_c)$$
$$d_{1,4} = 2n[k_{s_2} R_c J_n(k_{s_2} R_c) - J_n'(k_{s_2} R_c)]$$
$$d_{1,5} = 2n[k_{s_2} R_c Y_n'(k_{s_2} R_c) - Y_n'(k_{s_2} R_c)]$$
$$d_{2,1} = -k_{s_2} H_1^{(1)}(k_{s_2} R_c)$$
$$d_{2,2} = k_{s_2} R_c J_n'(k_{s_2} R_c)$$
$$d_{2,3} = k_{s_2} R_c Y_n'(k_{s_2} R_c)$$
$$d_{2,4} = n J_n(k_{s_2} R_c)$$
$$d_{2,5} = n Y_n(k_{s_2} R_c)$$
$$d_{3,2} = 2n[J_n(k_{s_2} R_c) - k_{s_2} R_c J_n'(k_{s_2} R_c)]$$
$$d_{3,3} = 2n[Y_n(k_{s_2} R_c) - k_{s_2} R_c Y_n'(k_{s_2} R_c)]$$
$$d_{3,4} = 2k_{s_2} R_c J_n'(k_{s_2} R_c) + (k_{s_2}^2 R_c^2 - 2n^2) J_n(k_{s_2} R_c)$$
$$d_{3,5} = 2k_{s_2} R_c Y_n'(k_{s_2} R_c) + (k_{s_2}^2 R_c^2 - 2n^2) Y_n(k_{s_2} R_c)$$
$$d_{4,2} = k_{s_2} R_c J_n'(k_{s_2} R_b)$$
$$d_{4,3} = k_{s_2} R_c Y_n'(k_{s_2} R_b)$$
$$d_{4,4} = n J_n(k_{s_2} R_b)$$
The elements of $a_4$ in (13) are given as follows:

$$a_1 = \frac{k_3^2}{k_4^2} J_n(k_3 R_c)$$

$$a_2 = k_1 R_c J_n'(k_1 R_c)$$

REFERENCES


