ESTIMATING CURRENT DENSITY IN THE HEART USING STRUCTURED AND UNSTRUCTURED COVARIANCE ANALYSIS

Aleksandar Jeremić and Arye Nehorai
The University of Illinois at Chicago
Chicago, Illinois, USA

ABSTRACT
The inverse problem of electrocardiography can be defined as the determination of the electrical activity of the heart from measurements of the body-surface electromagnetic field. The solution to this inverse problem may ultimately improve the ability to detect and treat cardiac diseases early. We present an algorithm for estimating the current density of the heart using ECG and magnetocardiography (MCG) sensor arrays. We model the electrical activity of the heart using current density represented by a set of deterministic and stochastic spatio-temporal basis functions. In order to solve the corresponding Fredholm equation we apply the element-free Galerkin method and compute the measurements as a function of the torso geometry and cardiac source. Then, we maximize the likelihood function to estimate the unknown parameters assuming a presence of spatially correlated Gaussian noise with unknown covariance matrix. Numerical examples illustrate the applicability of our results.

1. INTRODUCTION
It is well known that the inverse problem of electrocardiology is generally ill-posed since the intervening volume conductor (thorax) implies that the electromagnetic potentials measured on the surface are spatially very smooth projections of the electrical current sources [1]. In particular, the measured data is related to the desired image via a linear operator whose inverse is discontinuous. One way of making this ill-posed imaging problem better posed is to use a set of constraints on the solution which will sufficiently restrict the admissible class of solutions, so that the operator will have a continuous inverse [1]. In addition, these constraints can be related to the physiology and thus have physical meaning. We will impose these constraints implicitly in the formulation of the problem through our definition of the cardiac source.

We model the spatio-temporal density of the currents using a set of a priori known basis functions and unknown coefficients. In the first case, we assume that the current density is periodic in time which results in a deterministic model. In practice, however, the electrical activity of the heart is only asymptotically periodic. Therefore, to account for the possibly stochastic nature of the current density we also develop a stochastic model in which the coefficients corresponding to the spatio-temporal basis functions are random. Using these models and the element-free Galerkin (EFG) method [2], we derive expressions for the electromagnetic field on the torso surface and corresponding measurement models for an ECG/MCG sensor array. The EFG method is based on moving least-squares (MLS) interpolants for the test and trial functions and does not require any element connectivity data but can have the generality of a finite element (FE) method [2]. In addition, EFG does not seem to exhibit any volumetric locking even with the linear basis functions, and its rate of convergence can exceed that of the finite elements significantly. We also derive a parametric statistical models for the array’s measurements as a function of the basis functions coefficients in the presence of unknown spatially correlated Gaussian noise and derive a maximum likelihood (ML) estimator for the unknown parameters. Our use of spatially correlated noise should improve robustness with respect to perturbations such as noise, inhomogeneity, etc.

2. FORWARD MODEL
The forward problem in electro-cardiology is calculating the electric potential $\phi(r, t)$ and magnetic field $\mathbf{B}(r, t)$ at a location $r$ on the torso surface at a time $t$ from a given primary current distribution $\mathbf{J}(r', t)$ within the heart. We use a piecewise homogeneous torso model consisting of: the outer torso, inner torso, lungs, epicardium, and blood masses. Thus, we model the heart as a volume $G$ of $M = 7$ homogeneous layers separated by closed surfaces $S_i$, $i = 1, \ldots, M$. Let $\sigma_i^-$ and $\sigma_i^+$ be the conductivities of the layers inside and outside $S_i$, respectively. Denote by $G_i$ the regions of different conductivities, and by $G_{M+1}$ the region outside the torso, which behaves as an insulator. It has been shown that in the case of a piecewise homogeneous torso model and using quasi-
static assumption the electro-magnetic field for \( r \in G_k \) is given by [3]

\[
B(r, t) = B_0(r, t) + \frac{\mu_0}{4\pi} \sum_{i=1}^{M} (\sigma_i - \sigma_i^+): \cdot \int_{S_i} \phi(r', t) \frac{(r - r')}{||r - r'||^3} \times dS(r'),
\]

\[
B_0(r, t) = \frac{\mu_0}{4\pi} \int_G J(r', t) \times \frac{(r - r')}{||r - r'||^3} \, d^3r',
\]

(1)

\[
\frac{\sigma_i^- + \sigma_i^+}{2} = \phi_0(r)(\sigma_i^- - \sigma_i^+)+ + \frac{1}{4\pi} \sum_{i=1}^{M} (\sigma_i^- - \sigma_i^+) \int_{S_i} \phi(r', t) \frac{(r - r')}{||r - r'||^3} \, dS(r'),
\]

\[
\phi_0(r, t) = \frac{1}{4\pi} \int_G J(r', t) \cdot \frac{(r - r')}{||r - r'||^3} \, d^3r',
\]

(2)

where \( \mu_0 \) is the magnetic permeability of the vacuum. To solve the integral equations (1) and (2) we utilize the EFG method, see [2], where the function \( \phi(r, t) \) can be locally approximated by

\[
\tilde{\phi}(r, t) = \sum_{i=1}^{n_t} u_i(r, r') c_i(r', t),
\]

(3)

where \( n_t \) is the number of mesh basis functions, \( \{u_i(r, r')\} \) are basis functions, and \( \{c_i(r', t)\} \) are the coefficients. Details of the EFG approximation are given in [2], [7].

**Source Models**

In order to make the inverse problem better posed we propose constraining the current density to the function space spanned by a set of \textit{a priori} known spatio-temporal basis functions. This type of source constraint is particularly suitable for the current density since it has been shown that the electrical activity of the heart can be approximated well with the first few harmonics [1] (order of ten). We consider two source models: periodic which results in deterministic coefficients, and asymptotically periodic which results in random coefficients.

**Periodic Source Model**

Assume that the electrical activity of the cardiac sources is approximately periodic, i.e. \( J(r, t) = J(r, t + kT_c) \) where \( k = 0, 1, \ldots \) and \( T_c \) is the length of the heart cycle during which the measurements are obtained. In addition, we assume that the current density orientations are fixed in time. This constraint has been proposed in [5] in order to compensate for the non-physiological nature of the free-moment solutions.

Using this assumption, the current density can be written as a Fourier series

\[
J(r, t) = \sum_{i=0}^{2n_t} J_i(r) h_i(t)
\]

\[
J_i(r) = \begin{cases} 
J_i^+(r) & i = 0, \ldots, n_t \\
J_{i-n_t}(r) & i = n_t + 1, \ldots, 2n_t 
\end{cases}
\]

(4)

where \( n_t \) is number of temporal basis functions (Fourier harmonics), \( h_i(t) \) the \( i \)-th harmonic function, and \( \omega_c = 2\pi/T_c \).

We propose to model the spatial variability of \( J_i(r) \) using a set of \textit{a priori} known spatial basis functions \( \{\lambda_j(r)\}, j = 1, \ldots, n_s \) and unknown corresponding coefficients \( \{\theta_{i,j}\}, i = 0, \ldots, 2n_t \) which yields

\[
J(r, t) = \Lambda(r) \Theta h(t)
\]

(5)

where \( \Theta \) is a \( n_s \times 2n_t \) matrix of unknown source parameters \( \{\theta_{i,j}\}, \Lambda(r) = [\lambda_1(r) \ldots \lambda_{n_s}(r)] \) and \( h(t) = [h_0(t), \ldots, h_{2n_t}(t)]^T \).

It can be shown, see [8], that the electro-magnetic field is given by

\[
\phi(r, t) = \psi^T(r) H^{-1} Q \Theta h(t)
\]

(6)

\[
B(r, t) = (M(r) + Z(r) H^{-1} Q) \Theta h(t)
\]

(7)

\[
H_{i,j} = \int_G \partial_i(r) \partial_j(r) \, dr
\]

(8)

\[
M(r) = [\mu_1(r) \ldots \mu_{n_s}(r)]
\]

\[
Z(r) = [\eta_1(r) \ldots \eta_{n_s}(r)]
\]

(9)

\[
\eta_j(r) = \frac{\mu_0}{4\pi} \sum_{i=1}^{M} (\sigma_i^- - \sigma_i^+) \int_{S_i} \psi_j(r) \frac{(r - r')}{||r - r'||^3} \times dS(r').
\]

\[
Q = [\int_G v_1(r) \partial_1(r) dr \ldots \int_G v_{n_s}(r) \partial_{n_s}(r) dr]
\]

(10)

where \( \psi(r) \) is the mesh interpolation function, \( \{\partial_j(r)\} \) are the potential interpolation functions, see [8], and \( n \) is the number of mesh points.

**Asymptotically Periodic Source Model**

In reality the time length of the heart cycle changes with time i.e., the electrical activity of the heart is non-periodic. To account for this effect we assume that the period of the \( i \)-th cycle \( T_i \) is a random variable with a known mean \( T_c \) and the unknown variance \( \sigma_T^2 \). In addition we assume that the spatial amplitudes of the current density undergoes stochastic fluctuations. Let \( \tau_i \) and \( \tau_{i+1} \) denote the beginning and the end of the \( i \)-th cycle, respectively and let \( T_i = \tau_{i+1} - \tau_i \). Then the current density in the \( i \)-th cycle is given by

\[
J_i(r, t) = \begin{cases} 
J_0 \left( \frac{t - \tau_i}{T_c} \right) & \tau_i \leq t \leq \tau_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

(11)

where \( J_0(\cdot) \) is the reference current density. Applying the Fourier transform to (11) we obtain

\[
F\{J(r, t)\} = \sum_{i=1}^{n_s} T_i e^{-i\omega \tau_i} \int_0^{T_c} J_0(x) e^{-i\omega \frac{2\pi}{T_c} x} \, dx
\]

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As a result the stochastic signal (11) can be approximated by

\[ J(r, t) \approx \sum_{i=1}^{n_t} (c_i^2 + \Delta c_i^2) \sin(i(\omega_i + \Delta \omega_i)t). \] (12)

Using (12) and a derivation as in the periodic case we obtain

\[
\begin{align*}
\phi(r, t) &= \psi^T(r) H^{-1} Q(\Theta + \Delta \Theta)(h(t) + \Delta h(t)) \\
B(r) &= (\text{M}(r) + Z(r) H^{-1} Q(\Theta + \Delta \Theta))(h(t) + \Delta h(t)) \\
\Delta h(t) &= [\Delta h_0(t) \cdots \Delta_{2n_t}]
\end{align*}
\]

and \( \Delta \Theta \) is a random matrix of \([c_i^2, c_i^2]\). The above model is stochastic, i.e. the source parameters are both deterministic (\( \Theta \)) and random \( (\Delta \Theta, \Delta h(t)) \) variables.

### 3. STATISTICAL MODEL

#### Unstructured Covariance Model - Periodic Source

Assume that in the \( l \)th heart (measurement) cycle \((l = 1, \ldots, n_c)\), we obtain \( n \) temporal data vectors \( y_1, y_2, \ldots, y_n \) where \( y_k = y(t_k) \) is an ECG/MCG sensor array measurement vector. From here on we will refer to the matrix \( Y_1 = [y_1 \cdots y_n] \) as the cycle data matrix. The measurement model, (for detailed description see [8],) can then be written as

\[
Y = T \Theta H + E
\]

\[ T = \begin{bmatrix}
g^T(M(r_1) + Z(r_1)G^{-1}Q) \\
g^T(M(r_m) + Z(r_m)G^{-1}Q)
\end{bmatrix}. \] (13)

The columns \( \{e_k\} \) of the noise matrix \( E \) are assumed to be zero mean Gaussian distributed random vectors with unknown covariance \( \Sigma \), independent of time, and uncorrelated between different samples and cycles.

#### Structured Covariance Model - Asymptotically Periodic Source

We assume that the source properties change from cycle to cycle and use a subscript "c" to denote these variations. Let \( \Delta h_{k,l} = \Delta h(t_k) \) in the \( l \)th cycle. First, we assume that the product \( \| \Delta \Theta \Delta h_{k,l} \| \) is negligible. Next, we rewrite the measurement model as

\[
y_{k,l} = T(h_k^T \otimes I) \text{vec}(\Theta) + T(h_k^T \otimes I) \text{vec}(\Delta \Theta) + T \Delta h_{k,l} + e_{k,l}, \] (14)

where \( e_{k,l} \) is additive noise, the vec(\( \Theta \)) operator creates a column vector from the matrix \( \Theta \) by simply stacking the column vectors of \( \Theta \) below one another.

We use the following assumptions: vec(\( \Delta \Theta \)) is a zero mean Gaussian random vector with covariance \( \sigma_\delta^2 I_{(2n_t+1)q} \), \( \Delta h_{k,l} \) is a zero mean Gaussian random vector with covariance \( \sigma_h^2 I_{2n_t+1} \), thus uncorrelated from sample to sample, and from cycle to cycle, and \( e_{k,l} \) is a zero mean Gaussian random vector with covariance \( \sigma_e^2 I_m \) uncorrelated with both \( \Delta \Theta \) and \( \Delta h_{k,l} \). Using these assumptions the measurement vector becomes Gaussian distributed with mean

\[ T(h_k^T \otimes I) \text{vec}(\Theta) \] (15)

and covariance

\[ \Sigma = \sigma_h^2 \| h_k \|^2 TT^T + \sigma_e^2 \Theta \Theta^T T^T + \sigma^2 I. \] (16)

### 4. PARAMETER ESTIMATION

#### Unstructured Covariance Estimator

Stack all the cycle data matrices into one data matrix

\[ Y = [Y_1, Y_2, \ldots, Y_n] \]

where \( Y_1 \) is the cycle data matrix obtained in the \( l \)th cycle. Then, according to [6] the MLE of \( \Theta \) and \( \Sigma \) are

\[
\hat{\Theta} = (T^T S^{-1} A)^{-1} T^T S^{-1} Y H^T (H H^T)^{-1} \\
\hat{\Sigma} = S + (I - WS^{-1}) S_1 (I - WS^{-1})^T
\]

where

\[
\hat{R} = \frac{1}{n t_c} \sum_{i=1}^{n_c} Y_i Y_i^T \\
S = \hat{R} - S_1 \\
S_1 = \frac{1}{n t_c} Y H^T (H H^T)^{-1} H Y^T \\
W = T[T^T S^{-1} T]^{-1} T^T.
\] (18)

#### Structured Covariance Estimator

The main advantage of the unstructured model over the unstructured model is smaller number of parameters and thus better estimation efficiency. However, computing the MLE estimates requires implementation of a computationally expensive iterative algorithm. Furthermore, if the number of unknown parameters is large the algorithm may not converge. Since our interest lies primarily in \( \Theta \) we propose using an asymptotic MLE using the following steps: (i) compute the ordinary least squares estimate \( \hat{\Theta}_{\text{OLS}} = \arg \min \sum_{i=1}^{n_c} \| y_i - T(h_k^T \otimes I) \theta \|^2 \), where \( l = 1, \ldots, n_c, k = 1, \ldots, n \), and (ii) minimize the modified likelihood function assuming \( \Theta = \hat{\Theta}_{\text{OLS}} \). It can be shown that the iterative estimator converges to MLE, see [6].

### 5. NUMERICAL EXAMPLES

We conducted a numerical study using a real heart and simulating an 64-channel ECG/MCG system. We thank Professor MacLeod of the Department of Bioengineering, University of Utah for providing anatomically correct
The FE mesh of the torso and epicardial surface are illustrated in Figure 1. The magnetic-field component perpendicular to the sensor array surface ($B_z$) were measured. The configuration of the simulated sensors was chosen to coincide with the FE mesh points located on the torso surface front of the heart. The sampling rate of the sensor array was set to 5ms. We generated the current density using 10 harmonics and heart rate of 83 beats per minute. The spatial amplitudes were chosen randomly from (0; 800)pA interval. We model the temporal changes using $u(r, t) = h(t - \tau(r)) - h(t - \tau(r) - T(r))$ where $h(\cdot)$ is the Heaviside function, $\tau(r)$ is the activation time at the location $r$ (see [4]), and $T(r)$ is the time length of ionic current pulse. We simulated 100 runs, each consisting of $n = 10$ cycles. We generated spatially correlated noise using 10 random dipoles uniformly distributed on the torso surface. For each dipole we assumed that its two tangential moment components were uncorrelated and distributed as $N(0, \sigma_n^2)$. For $\sigma_n^2 = 1pA \cdot m$, the total noise standard deviation was approximately 60fT. To model the stochastic source we generated the “jittery” current by adding spatial perturbations of $\Delta J_0(r, t)$ at each mesh point and time sample by $\Delta J \sim N(0, \sigma^2_t)$ where $\sigma^2_t = 0.1pA$, thus allowing only small perturbations. In addition, we perturb the length of each cycle by $\Delta T_c \sim N(0, \sigma^2_c)$ and set $\sigma^2_c = 10ms$. In Fig. 2 we compare the average mean-square error (MSE) for various algorithms as functions of $\sigma_n$ for the periodic source. As expected the unstructured MLE significantly outperforms the OLS because OLS does not account for the spatial correlation of noise. The structured MLE also shows significant improvement in the performance compared with OLS. However, the unstructured MLE has superior performance over the structured MLE since it does not assume a structured covariance. In Figure 3 we compare the MSE of the structured and the unstructured MLE as functions of the total noise $\sigma$ for the stochastic source. As expected, the structured MLE outperforms the unstructured MLE because it uses adequate number of the unknown covariance parameters.

REFERENCES


