Performance Analysis of Direction Finding with Large Arrays and Finite Data
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Abstract—This paper considers analysis of methods for estimating the parameters of narrow-band signals arriving at an array of sensors. This problem has important applications in, for instance, radar direction finding and underwater source localization. The so-called deterministic and stochastic maximum likelihood (ML) methods are the main focus of this paper. A performance analysis is carried out assuming a finite number of samples and that the array is composed of a sufficiently large number of sensors. Several thousands of antennas are not uncommon in, e.g., radar applications. Strong consistency of the parameter estimates is proved, and the asymptotic covariance matrix of the estimation error is derived. Unlike the previously studied large sample case, the present analysis shows that the accuracy is the same for the two ML methods. Furthermore, the asymptotic covariance matrix of the estimation error coincides with the deterministic Cramér-Rao bound. Under a certain assumption, the ML methods can be implemented by means of conventional beamforming for a large enough number of sensors. We also include a simple simulation study, which indicates that both ML methods provide efficient estimates for very moderate array sizes, whereas the beamforming method requires a somewhat larger array aperture to overcome the inherent bias and resolution problem.

I. INTRODUCTION

SENSOR array signal processing has been an active research area for more than three decades. The generic problem is to determine unknown signal parameters from simultaneous measurements of spatially distributed sensors. Several important applications have been reported, for example, interference rejection in radar and radio communication systems and source localization using underwater sonar arrays.

A vast number of methods have been proposed for solving the estimation problem; see e.g. [1]–[3] and the references therein. Asymptotic results for several estimators have recently appeared in the literature, assuming the number of the samples (snapshots), $N$, to be large, [4]–[7]. In particular, it has been shown that the maximum likelihood method based on a stochastic, Gaussian model of the signal waveforms is asymptotically (for large $N$) efficient, i.e., the estimation error covariance attains the corresponding Cramér-Rao bound (CRB). On the other hand, the ML estimator based on a deterministic model of the signal waveforms is not efficient. This is due to the fact that the number of estimated parameters in the deterministic model increases with increasing sample size, whereas the number of parameters in the stochastic model remains fixed.

In many real-time applications, the large sample results are of little use due to a limited data collection time, a nonstationary scenario, and/or the effect of damped signal waveforms. To obtain accurate parameter estimates in these cases, it is generally necessary to employ an array with a large number $m$ of sensors. Arrays of up to 20 000 antennas are not uncommon in, for example, radar systems, see, e.g., p. 11 of [8]. In applications with such a large number of sensors, an initial beamforming is usually performed, and the data is processed in “beamspace”; see, e.g., [9]. It has recently been shown that the beamspace transformation (i.e., the “bank of beamformers”) can be chosen such that the signal parameter estimation based on the reduced dimension data is as accurate as that obtained using all sensor outputs [10]. Hence, the present analysis is also applicable to beamspace data, provided such an optimal transformation is used.

An analysis of the various estimators in the case $m \gg 1$ is different from the previously studied “large $N$” case. It has been shown earlier that the stochastic and deterministic ML methods are asymptotically equivalent when both $N$ and $m$ are large; see [6]. Herein, the equivalence is verified for arbitrary $N$, and the covariance matrix of the estimation error is shown to attain the CRB under the deterministic signal model.

Since the computational requirements of the ML techniques increase significantly with increasing $m$, computationally more efficient alternatives are of great interest. Under certain assumptions on the array geometry, the Fisher information matrix (FIM) is asymptotically diagonal, and it is shown that the ML estimators can be implemented by means of the conventional beamforming method without impairing the asymptotic efficiency. This is, in fact, possible even in the presence of completely coherent signal waveforms.

The remainder of the paper is organized as follows: In Section II, the sensor array problem is formulated, and the mathematical preliminaries are presented. A brief description of the estimation methods to be analyzed is also given. In Section III, we verify that the estimates converge to the true values as the number of sensors tend to infinity. Section IV concerns the asymptotic accuracy of the estimates. A simple example is presented in Section V to illustrate the applicability of the asymptotic results to arrays of finite size.
II. ESTIMATION PROBLEM

This section formulates the direction-of-arrival (DOA) estimation problem and briefly describes the methods under consideration.

A. Problem Formulation

Consider an array of \( m \) sensors having arbitrary positions and response characteristics. Impinging on the array are the waveforms of \( d \) far-field point sources, where \( d < m \). Under the narrow-band assumption, the array output \( x(t) \) is modeled by the following equation:

\[
x(t) = A(\theta)s(t) + n(t).
\]

The columns of the \( m \times d \) matrix \( A \) are the array propagation vectors \( a(\theta_i), i = 1, \ldots, d \). These vectors are functions of the DOA's and they model the array response to a unit waveform from the direction \( \theta_i \). The signal parameters are collected in the parameter vector \( \theta^T = [\theta_1, \ldots, \theta_d] \). We will often refer to \( \theta_0 \) as the vector of true DOA's. The \( d \)-vector \( s(t) \) is composed of the complex emitter waveforms received at time \( t \), and the \( m \)-vector \( n(t) \) accounts for additive measurement noise as well as modeling errors. The array output is sampled at \( N \) distinct time instants, and the data matrix is formed

\[
X_N = A(\theta)S_N + N_N
\]

where \( X_N = [x(1), \ldots, x(N)], S_N = [s(1), \ldots, s(N)] \), and \( N_N = [n(1), \ldots, n(N)] \). Based on these measurements, the problem of interest is to determine the DOA's of all emitters. The number of signals \( d \) is assumed to be known.

The main interest herein is to ascertain the accuracy of the maximum likelihood (ML) estimates of \( \theta \). Two ML approaches have been studied in the literature. In the so-called deterministic model, no restrictions are put on the signal waveforms, i.e., they are modeled as arbitrary sequences [3], [11]. An alternative is the stochastic model, where the signals are assumed to be stationary Gaussian random processes. In both models, the noise term \( [n(t)] \) is assumed to be Gaussian. The noise has zero mean and is spatially and temporally white, i.e.

\[
E[n(t)n^*(s)] = \sigma^2 I_{t-s}
\]

where \( \delta(t) \) is the Kronecker delta, and \( (\cdot)^* \) is the complex conjugate transpose. Both methods are applicable regardless of the actual distribution of the signal waveforms, although they provide maximum likelihood estimates only under their respective assumption. We will study the accuracy of the estimates under the deterministic model. This is, perhaps, the more natural model to use in applications where only a small number of snapshots is available.

A crucial assumption for most DOA estimation methods is that the functional form of \( a(\cdot) \) is available to the user. For simplicity, it will be assumed that the sensors are omnidirectional and scaled so that \( |a(\theta)| = \sqrt{m} \). If is further assumed that \( a(\theta) \) has bounded third derivatives with respect to \( \theta \).

To enable unique identification of the DOA's, it is usually assumed that the array manifold has no ambiguities [1], i.e., that for distinct parameters \( \theta_1, \ldots, \theta_m \), the matrix \( [a(\theta_1), \ldots, a(\theta_m)] \) has full rank. Since the dimension of this matrix increases in the present case, a slightly different condition is required.

**Assumption 1:** Let \( \Theta_d \) denote the set of \( d \)-dimensional parameter vectors of interest. Assume that there exists an \( \epsilon > 0 \) and an \( M \geq 2d \) such that for all \( \eta \in \Theta_{2d} \) and \( m \geq M \), it holds that

\[
\frac{1}{m} A^*(\eta)A(\eta) \geq \epsilon I.
\]

The matrix inequality \( X \geq Y \) means that \( X - Y \) is positive semidefinite.

Note that (4) implicitly assumes that we confine ourselves to a certain minimum resolution. In other words, if \( \theta \in \Theta_d \), then there is a \( \delta > 0 \) such that \( |\theta_k - \theta_l| \geq \delta, k \neq l \). For \( d = 1, (4) \) has a simple interpretation. It means that the level of the normalized beampattern at any fixed DOA \( \theta \) must remain lower than the mainlobe by a fixed amount for all \( m \). For most types of arrays, the above mentioned analogy can be made also for \( d > 1 \). However, in the general case, the assumption of a bounded beampattern does not guarantee that (4) is satisfied since the array may have so-called "high-order ambiguities" [1]. The idea of the analysis and the requirement of having a large number of sensors, rather than just a large array aperture, is perhaps best illustrated in an example.

**Example 1—Large Aperture but Fixed \( m \):** Consider an array of \( m = 3 \) sensors with elements located along the \( y \) axis in the \( xy \) plane at the coordinates \( 0, \frac{\lambda}{2} \) and \( \frac{3\lambda}{2} \). Suppose a single far-field source is located at \( \theta_\delta = 0^\circ \) (along the \( x \) axis). The first two sensors provide sufficient information for the unique estimation of \( \theta \). For this reason, one might guess that by increasing \( \Delta \) (i.e., increasing the aperture of the array), it is possible to get arbitrarily good estimates for fixed \( N \) and signal-to-noise ratio (SNR). Indeed, it is easy to show that the Cramér-Rao lower bound tends to zero as \( \Delta \rightarrow \infty \), hence supporting the claim. However, consider Fig. 1, which depicts the normalized beampattern \( [a^*(\theta)a(\theta_\delta)]^2/m^2 \) for \( \theta_\delta = 0^\circ \) and \( \Delta = 20 \). As is well known, a large aperture but a small number of sensors results in a beampattern with a large number of high sidelobes. Given a fixed number \( N \) of noisy observations of the array output, the beamforming criterion (which coincides with maximum likelihood for \( d = 1 \)) is a distorted version of the beampattern. Thus, the global maximum could occur at any of the "false" peaks of the criterion rather than at the main lobe. Consistent estimation of the DOA's is therefore not possible when only the aperture is increased without bound. It is necessary to reduce the effect of the noise by some form of averaging. This can be done by increasing the collection time \( N \) or, as shown herein, by increasing the number of sensors \( m \).

A full understanding of the actual attainable estimation error variance in Example 1 requires the use of the global properties of the likelihood function rather than only the local second-order properties exploited by the CRB. Such an analysis falls beyond the scope of the present study.

Note that Assumption 1 is trivially satisfied for a uniform linear array (ULA) since the steering vectors are asymptotically (for large \( m \)) orthogonal. When analyzing the beamform-
Fig. 1. Beam pattern for three-element sparse array with aperture equal to 10 wavelengths.

ing method, the following stronger assumption appears to be necessary.

**Assumption 2:** Array response vectors corresponding to different DOA’s are orthogonal for large m, i.e.

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{N} \mathbf{a}^*(\theta_i) \mathbf{a}(\eta) = \begin{cases} 
1, & \theta = \eta \\
0, & \theta \neq \eta 
\end{cases}
\]  

(5)

Note that Assumption 2 implies Assumption 1.

**Remark 1:** A few comments on the validity of the data model are in order. Increasing the number of sensors normally means that in addition, the physical size of the array is increased. However, the narrow-band assumption constrains the diameter \( R \) of the array to \( R \ll c/B \), where \( c \) is the speed of propagation, and \( B \) is the signal bandwidth. In addition, too large an \( R \) means that the far-field assumption is violated. The latter is, in principle, not a problem—we may simply let the steering vectors be functions of both range and DOA in such a case. If, on the other hand, \( m \) is increased while keeping \( R \) fixed, the array becomes increasingly dense. From physical reasons, the covariance matrix of the noise must then become increasingly ill conditioned, thus preventing a prewhitening of the noise.

Finally, since the array propagation vectors have length \( |\mathbf{a}(\theta)| = \sqrt{m} \), where \( m \) is increased in the analysis, the total received signal energy increases without bound. This assumption also limits the physical size (the signals must be in the far field) and the density of the array. It is understood that the results to be presented are applicable only to cases where a fixed, but “large enough,” number of sensors are available, subject to the above-mentioned constraints.

**B. Estimation Methods**

A systematic approach to many parameter estimation problems is the maximum likelihood (ML) method. This technique requires a probabilistic setup for the measurements. If the emitter signals are modeled as unknown deterministic quantities, the observation process is distributed as \( \mathbf{x}(t) \in N(\mathbf{Aa}(t), \sigma^2 \mathbf{I}) \). The unknown parameters in this model are \( \theta^T = [\theta_1, \ldots, \theta_d] \) as well as \( \sigma(1), \ldots, \sigma(N), \sigma^2 \). Introduce the pseudoinverse of \( A \) and the orthogonal projector onto the nullspace of \( A^* \) by

\[
A^\dagger(\theta) = (A^* A)^{-1} A^*
\]

\[
P_{\mathbf{A}}^\perp(\theta) = I - AA^\dagger = I - P_{\mathbf{A}}.
\]

The deterministic maximum likelihood (DML) estimate of \( \theta \) is obtained as [3], [11]

\[
\hat{\theta}_{DML} = \arg \min_{\theta} \text{Tr}(P_{\mathbf{A}}^\perp(\theta) \hat{\mathbf{R}})
\]

(8)

where \( \text{Tr} \{ \cdot \} \) denotes the trace and where \( \hat{\mathbf{R}} \) is the sample covariance matrix

\[
\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=1}^{N} \mathbf{x}(t) \mathbf{x}^*(t).
\]

(9)

A different ML estimate is obtained by instead modeling the emitter signals as stationary, temporally white Gaussian random processes with covariance \( \mathbf{S} \). The observation process is, in this case, distributed as \( \mathbf{x}(t) \in N(0, A \mathbf{S} A^* + \sigma^2 \mathbf{I}) \).

The unknown parameters in this so-called stochastic model are \( \theta, \mathbf{S} \), and \( \sigma^2 \). For \( N \geq d \), the stochastic maximum likelihood (SML) estimate has been shown [2] to be

\[
\hat{\theta}_{SML} = \arg \min_{\theta} \left| \mathbf{A} \hat{\mathbf{S}}(\theta) \mathbf{A}^* + \sigma^2(\theta) \mathbf{I} \right|
\]

(10)

\[
\hat{\mathbf{S}}(\theta) = A^\dagger(\theta) \mathbf{R} - \sigma^2(\theta) \mathbf{I} A^\dagger^*
\]

(11)

\[
\hat{\sigma}^2(\theta) = \frac{1}{m - d} \text{Tr}(P_{\mathbf{A}}^\perp \hat{\mathbf{R}})
\]

(12)

where \( | \cdot | \) denotes the determinant.

The exact ML criteria presented above both suffer from the requirement of a nonlinear multidimensional optimization in order to calculate the estimates. This is particularly prominent when \( m \) is large since the computational cost is at least proportional to \( m^2 \) [12]. In general, the ML criteria have several local minima, and it is difficult to guarantee convergence to the global minimum. To overcome this difficulty, several suboptimal techniques have been proposed. Perhaps the most natural of these is the traditional “delay-and-sum” beamforming method. This technique can be derived from the DML criterion (8) by setting \( \mathbf{A}(\theta) = \mathbf{a}(\theta) \) and searching for \( d \) distinct minima in the so-obtained 1-D criterion. The beamforming criterion function is thus

\[
\text{Tr}(P_{\mathbf{A}}^\perp(\theta) \hat{\mathbf{R}}) \propto -\frac{\mathbf{a}^*(\theta) \hat{\mathbf{R}} \mathbf{a}(\theta)}{\mathbf{a}^*(\theta) \mathbf{a}(\theta)} \propto -\mathbf{a}^*(\theta) \hat{\mathbf{R}} \mathbf{a}(\theta).
\]

(13)

Under the assumption that \( |\mathbf{a}(\theta)| \) is independent of \( \theta \), it is easy to show that (13) is in fact also proportional to the SML criterion (10) when \( d = 1 \).

**III. CONVERGENCE ANALYSIS**

The analysis of the described methods for large \( m \) is different from the large \( N \) case in some important respects. A complication results from the fact that the steering vectors may be orthogonal for \( m \) tending to infinity. This is indeed the case for the ubiquitous ULA. The asymptotic orthogonality between steering vectors causes a discontinuity in the limiting
criterion function. Consider, for example, the beamforming method under Assumption 2. It is easy to show that
\[
\lim_{m \to \infty} \frac{1}{m} \mathbf{a}^*(\theta) \hat{R}_m(\theta) = \left\{ \begin{array}{ll}
\hat{S}_i, & \theta = \theta_i \\
0, & \theta \neq \theta_i \end{array} \right.
\]
where \( \hat{S}_i = \frac{1}{N} \sum_{t=1}^N |s_i(t)|^2 \) and where \( \theta_i, i = 1, \ldots, d \) represent the true DOA’s. Since the limiting criterion function is discontinuous, the convergence of the sequence of cost functions cannot be uniform in \( \theta \). This makes it difficult to establish consistency of the estimates since these do not necessarily converge to the minimizing arguments of the limiting criteria. To overcome this, we shall use a convergence result that will be useful for proving consistency despite the lack of uniform convergence.

**Lemma 1:** Let \( \hat{\theta}_m \) be obtained by solving
\[
\hat{\theta}_m = \arg \min_{\theta} V_m(\theta)
\]
where the criterion function consists of a “signal” and a “noise” term
\[
V_m(\theta) = f_m(\theta) + \varepsilon_m(\theta).
\]
Assume that these terms satisfy the following conditions:
i. \( \lim_{m \to \infty} \varepsilon_m(\theta) = 0 \) uniformly in \( \theta \).
ii. There exists a point \( \theta_0 \) such that for any \( \delta > 0 \), we can find \( \epsilon > 0 \) and \( M \) satisfying
\[
f_m(\theta) \geq f_m(\theta_0) + \epsilon
\]
for all \( \theta \) and \( m \) satisfying \( |\theta - \theta_0| > \delta \) and \( m > M \).

Then
\[
\lim_{m \to \infty} \hat{\theta}_m = \theta_0.
\]

**Proof:** Select \( \delta > 0 \) arbitrarily. Using ii) and (14), we can find constants \( \epsilon > 0 \) and \( M_1 \) such that
\[
V_m(\theta) - V_m(\theta_0) \geq \epsilon + \varepsilon_m(\theta) - \varepsilon_m(\theta_0)
\]
for all \( \theta \) with \( |\theta - \theta_0| > \delta \) and \( m > M_1 \). By ii), there exists an \( M_2 > M_1 \) with
\[
\sup_{\theta \in \Theta} |\varepsilon_m(\theta)| < \epsilon / 2
\]
for all \( m > M_2 \). Hence
\[
V_m(\theta) - V_m(\theta_0) \geq \epsilon - \epsilon / 2 = 0
\]
for all \( m > M_2 \) and \( |\theta - \theta_0| \geq \delta \). However, \( V_m(\hat{\theta}_m) \leq V_m(\theta_0) \), implying \( |\hat{\theta}_m - \theta_0| < \delta \) for all \( m > M_2 \). Since \( \delta \) is arbitrary, the result follows.

**A. Deterministic ML**

Based on the above, let us first establish strong consistency of the DML estimate.

\[
\sup_{\theta \in \Theta} \frac{1}{m} \text{Tr}(\hat{P}_A A_0 \hat{Z}) \leq \text{Tr} \left( \frac{\hat{Z} A_0}{m} \right) + \sup_{\theta \in \Theta} \text{Tr} \left( \frac{\hat{Z} A}{m} \times (A^* A)^{-1} \times \frac{A^* A_0}{m} \right) \leq C \sup_{\theta} \left\| \frac{\hat{Z} A}{m} \right\|_F.
\]

**Theorem 1:** Under Assumption 1, the DML estimate of \( \theta \) converges w.p.1 to the true parameter vector \( \theta_0 \) as \( m \to \infty \).

**Proof:** Introduce
\[
\hat{S} = N^{-1} \sum_{t=1}^N s(t) s^*(t) \quad \hat{Z} = N^{-1} \sum_{t=1}^N s(t) n^*(t)
\]
and subtract the \( \theta \)-independent term \( \text{Tr}(\hat{Z})/m \) from the DML criterion function (8) to yield
\[
V_m(\hat{\theta}) = \frac{1}{m} \text{Tr}(\hat{P}_A \hat{R}) - \frac{1}{m} \text{Tr}(\hat{Z}).
\]

By (9) and (21), the sample covariance matrix can be expressed as
\[
\hat{R} = A(\theta_0) \hat{S} A^*(\theta_0) + A(\theta_0) \hat{Z} + \hat{Z} A^*(\theta_0) + \hat{Z}.
\]

Hence, we can write \( V_m(\hat{\theta}) = f_m(\hat{\theta}) + \varepsilon_m(\theta) \), where
\[
f_m(\hat{\theta}) = \frac{1}{m} \text{Tr}(\hat{P}_A A_0 \hat{S} A^*(\theta_0))
\]
\[
\varepsilon_m(\theta) = \frac{1}{m} \text{Tr}(\hat{P}_A A_0 \hat{Z}) + \frac{1}{m} \text{Tr}(\hat{P}_A \hat{Z}).
\]

Let us first verify the condition i) of Lemma 1. Consider the first term of (25). Using the short-hand notation \( A_0 = A(\theta_0) \) and \( A = A(\theta) \), we have (26), which appears at the bottom of the page, for some constant \( C \) (independent of \( m \)). The first inequality in (26) is the triangle inequality, and in the second inequality, we used the Cauchy-Schwartz inequality. Theorem 1, and the fact that \( |a(\theta)|^2 = m \). The matrix appearing on the right-hand side of (26) can be expressed as
\[
\langle \hat{Z} A(\theta_0)/m \rangle_{ij} = \langle \frac{1}{N} \sum_{t=1}^N s(t) \times \frac{1}{m} \sum_{k=1}^m n_k(t) A_{kj}(\theta) \rangle.
\]

By Theorem 4.2.3 of [13] and using \( |a(\theta)|^2 = m \), there is a constant \( C < \infty \) such that
\[
\sup_{\theta \in \Theta} \frac{1}{m} \text{Tr}(\hat{P}_A A_0 \hat{S} A_0^*) \leq \frac{1}{m} \| A_0 L + AT \|_F^2
\]
where \( L \) and \( T \) are defined by the relations
\[
\hat{S} = LL^*
\]
\[
T = -A^* A_0 L.
\]
Let \( k \) denote the number of components of \( \theta \) that are not equal to any component of \( \theta_0 \). If \( |\theta - \theta_0| > \delta > 0 \), we must have \( 1 \leq k \leq d \). Assume, without loss of generality, that the unique DOA’s in \( \theta_0 \) and \( \theta \) are \( \theta_0_1, \ldots, \theta_0_k \) and \( \theta_1, \ldots, \theta_k \), respectively. Let \( \mathbf{t}_i^T \) and \( \mathbf{t}_j^T \) denote the rows of \( \mathbf{L} \) and \( \mathbf{T} \) and introduce the matrices

\[
\mathbf{Q} = \begin{bmatrix} t_1, \ldots, t_k, t_j, \ldots, t_{k+1} + t_{k+1}, \ldots, t_d \end{bmatrix}^T
\]

\[
\mathbf{B} = [a(\theta_0_1), \ldots, a(\theta_0_k), a(\theta_1), \ldots, a(\theta_k)].
\]

We can then rewrite (29) as

\[
f_m(\theta) = \frac{1}{m} \text{Tr}(\mathbf{BQ}_Q^* \mathbf{B})
\]

\[
\geq \epsilon \text{Tr}(\mathbf{Q}_Q^* \mathbf{Q}_Q)
\]

\[
= \epsilon \mathbf{S}_{11} > 0 \Rightarrow f_m(\theta) = \epsilon(\theta).
\]

In the last expression, \( \mathbf{S}_{11} = |t_i|^2 \) denotes the upper left element of \( \mathbf{S} \), which is nonzero by assumption, and in the first inequality, we have used Assumption 1. Now, (34) holds for all \( m \) (sufficiently large), which shows that condition ii) of Lemma 1 is satisfied with \( \theta = \theta_0 \). The assertion of the Theorem follows.

B. Stochastic ML

It is difficult to analyze the stochastic ML method in (10) due to the increasing dimension \((m \times m)\) of the parameterized covariance matrix. We will make use of the determinant identity \([AB] = [BA]\) to rewrite the criterion function in a form that is easier to analyze and, in fact, simplifies numerical evaluation of the criterion:

\[
\log |\mathbf{S}(\theta)\mathbf{X} + \tilde{\sigma}^2 (\theta)I| = \log \tilde{\sigma}^{m^2}(\theta) |\mathbf{X}^2(\theta)\mathbf{A}^* + I|
\]

\[
= \log \tilde{\sigma}^{m^2}(\theta) |\mathbf{X}^2(\theta)\mathbf{A}^* + I|
\]

\[
= \log \tilde{\sigma}^{m^2}(\theta) |\mathbf{S}(\theta)\mathbf{X}^2 + \tilde{\sigma}^2 (\theta)I|.
\]

Inserting (11) and (12) into the expression above leads, after some manipulations, to the normalized cost function

\[
V_m(\theta) = \log \frac{1}{m-d} \text{Tr}(\mathbf{P}_\mathbf{X}^2 \mathbf{R}) + \frac{1}{m-d} \log |\mathbf{A}^* \mathbf{R}^2| - \frac{1}{m-d} \log |\mathbf{A}^* \mathbf{R}^2|.
\]

It is clear that the determinant of \( \mathbf{A}^* \mathbf{R}^2 \) is zero whenever \( N < d \). This is why \( N \geq d \) must be assumed for the concentrated form (10) of the SML criterion to be valid. Using (36), we can establish the strong consistency of the SML estimate.

\[\text{Theorem 2: Let Assumption 1 hold, and assume in addition that } N \geq d. \text{ The SML DOA estimate then converges w.p.1 to } \theta_0 \text{ as } m \text{ tends to infinity.}\]

\[\text{Proof: Write } V_m(\theta) = f_m(\theta) + \epsilon_m(\theta), \text{ where } f_m(\theta) = \log \frac{1}{m-d} \text{Tr}(\mathbf{P}_\mathbf{X}^2 \mathbf{R}) \text{ and } \epsilon_m(\theta) = \frac{1}{m-d} \log |\mathbf{A}^* \mathbf{R}^2| - \frac{1}{m-d} \log |\mathbf{A}^* \mathbf{R}^2|.
\]

Obviously, \( f_m(\theta) \) is minimized by the DML estimate. It follows from the proof of Theorem 1 and from (20) that \( f_m(\theta) \) satisfies condition ii) of Lemma 1 with \( \mathbf{R} = \theta_0 \). Furthermore, it is easy to see that for large \( m \)

\[
\sup_{\theta} |\mathbf{A}^* \mathbf{R}^2| \leq C m^2 \text{ w.p.1}
\]

\[
\sup_{\theta} |\mathbf{A}^* \mathbf{R}^2| \leq C m
\]

implying that \( \epsilon_m(\theta) \) converges to zero uniformly in \( \theta \). Application of Lemma 1 now establishes Theorem 2.

C. Beamforming

As explained in Section II-B, the beamforming method de-couples the \( d \)-dimensional search required by the ML methods into the search for the \( d \) smallest separated local minima in a 1-D version of the same criterion. This is reasonable only if steering vectors corresponding to distinct DOA’s are approximately orthogonal. Thus, the consistency of the beamforming method requires the condition provided by Assumption 2.

\[\text{Theorem 3: If Assumption 2 holds, the beamforming estimates converge to the true DOA’s w.p.1 as } m \rightarrow \infty.\]

\[\text{Proof: By (13), the beamforming estimates are the smallest separated local minima of the normalized criterion } V_m(\theta) = \frac{-\mathbf{a}^*(\theta) \mathbf{R} \mathbf{a}(\theta)}{\mathbf{a}^*(\theta) \mathbf{R} \mathbf{a}(\theta)} = \frac{-\mathbf{a}^*(\theta) \mathbf{R} \mathbf{a}(\theta)}{m^2}
\]

\[\text{Using (21), this can be written as } V_m(\theta) = f_m(\theta) + \epsilon_m(\theta), \text{ with } \theta = \theta_0 \text{ and } \epsilon_m(\theta) = \frac{\mathbf{a}^*(\theta) \mathbf{Z} \mathbf{a}(\theta)}{m^2}
\]

\[\text{The proof that } \epsilon_m(\theta) \text{ converges to zero uniformly in } \theta \text{ is analogous to the corresponding result for the DML criterion. Condition ii) of Lemma 1 holds since by Assumption 2}
\]

\[\lim_{m \rightarrow \infty} f_m(\theta) = \left\{ \begin{array}{ll}
\mathbf{S}_{ii} < 0, & \text{if } \theta = \theta_i \\
0, & \text{otherwise}
\end{array} \right.
\]

Thus, \( \hat{\theta} \) can converge to any of the true DOA’s but not to any other value, which concludes the proof.

Note from the above proof that Theorem 3 does not require the emitter covariance matrix (or sample covariance matrix) to be of full rank. Thus, under the assumptions of the theorem, the beamforming method provides consistent parameter estimates, even in the presence of coherent signals. Recall the well-known fact that the peaks of the periodogram give consistent estimates (as the data length tends to infinity) of the frequencies of superimposed sinusoids in noise. This problem corresponds to the one studied herein with \( N = 1 \) and a ULA of \( m \) sensors, where \( m \rightarrow \infty \). Clearly, the issue of signal correlation is irrelevant when \( N = 1 \) since the emitter sample covariance matrix is then always of rank one.

IV. ASYMPTOTIC EFFICIENCY

A. The ML Methods

It is well known that if an ML estimator provides consistent estimates (of all its unknown parameters), it is also asymptoti-
cally efficient, i.e., the asymptotic covariance of the parameter estimates coincides with the CRB. In the deterministic signal model, the unknown parameters are $\theta, S_N$ and $\sigma^2$. We have already shown that $\hat{\theta}_{DML}$ is consistent. For the signal waveforms, we have

$$
\{\tilde{S}_N\}_DML = A^\dagger(\hat{\theta}_{DML})X_N
$$

$$
= A^\dagger(\hat{\theta}_{DML})A(\theta_0)S_N + A^\dagger(\hat{\theta}_{DML})N - S_N
$$

(45)

since $m(A^*A)^{-1}$ is bounded, and $A^*N/m$ tends to zero w.p.1. The DML estimate of the noise variance is well known to be

$$
\hat{\sigma}^2_{DML} = \frac{1}{m} \text{Tr}\left(P_{\hat{\theta}_{DML}}N\right).
$$

(46)

Using that $\hat{\theta}_{DML} \to \theta_0$ w.p.1, it is straightforward to show that also $\hat{\sigma}^2_{DML}$ is consistent. Having established the consistency of all unknown parameters in the DML formulation, we can apply the general theory of ML estimation to give the following theorem.

**Theorem 4:** Let Assumption 1 hold. Then, as $m \to \infty$, the estimation error $\hat{\theta}_{DML} - \theta_0$ has a limiting Gaussian distribution with zero mean and covariance matrix equal to the CRB under the deterministic signal model. We use the notation

$$
(\hat{\theta}_{DML} - \theta_0) \in \mathcal{A}N(0, \text{CRB}).
$$

(47)

In [5] and [14], the following explicit expression for the CRB is derived:

$$
\text{CRB} = \frac{\sigma^2}{2N} \left[\text{Re}\left\{D^*P_{\hat{\theta}_{DML}}D\right\} \odot S^T\right]^{-1}
$$

(48)

where $\odot$ means elementwise multiplication, and

$$
D = \left[\frac{\partial D}{\partial \theta_0} \bigg|_{\theta = \theta_0}, \ldots, \frac{\partial D}{\partial \theta_0} \bigg|_{\theta = \theta_0}\right].
$$

(49)

**Proof:** A proof of asymptotic normality and efficiency for the case of independent and identically distributed observations can be found in Theorem 4.1, Part (ii) of [15]. The extension to our case is straightforward using a corresponding extension of the central limit theorem; see, e.g., Lemma 9.4.2 of [16].

Given (36) and the fact that the second term converges to zero uniformly in $\theta$, it is tempting to guess that the SML and DML methods are asymptotically identical. However, a standard Taylor series expansion (see, e.g., [7]) shows that this reasoning is correct only if it is also verified that the first derivative of the "noise" term (38) is negligible compared with the derivative of the first term for large $m$.

**Theorem 5:** Let the assumptions of Theorem 2 hold, and assume in addition that the signals are noncoherent so that $\hat{S} > 0$. Then, $\hat{\theta}_{DML}$ and $\hat{\theta}_{SML}$ are asymptotically equivalent for large $m$.

**Proof:** Consider first (37). We have

$$
\frac{\partial}{\partial \theta_i} f_m(\theta) \bigg|_{\theta = \theta_0} = \frac{2}{\sigma^2} \hat{\sigma}_i^2 \approx \frac{1}{\sigma^2} \hat{\sigma}_i^2
$$

(50)

where

$$
\hat{\sigma}^2 = \hat{\sigma}^2(\theta_0) = \frac{1}{m - d} \text{Tr}\{P_{\hat{\theta}_{DML}}N\}
$$

(51)

and where $\hat{\sigma}_i^2 = \partial^2 \hat{\theta}_i / \partial \theta_i \partial \theta_0$ evaluated at $\theta_0$. In (50), the easily checked fact that $\hat{\sigma}^2(\theta_0) \sim \sigma^2$ as $m \to \infty$ is used. The derivative of the projection matrix is [7], [17]

$$
\frac{\partial}{\partial \theta_i} P_{\hat{\theta}_{DML}} = -\frac{\partial}{\partial \theta_i} P_{\hat{\theta}_{DML}} = P_{\hat{\theta}_{DML}} A_i A_i^\dagger + A_i^\dagger A_i^\dagger P_{\hat{\theta}_{DML}}
$$

(52)

where the notation $A_i = \partial A / \partial \theta_i$ is used. Inserting this in (51) gives

$$
\hat{\sigma}_i^2 = -\frac{2}{m - d} \text{Re}\left[\text{Tr}\{P_{\hat{\theta}_{DML}} A_i A_i^\dagger\}\right]
$$

$$
= -\frac{2}{m - d} \text{Re}\left[\text{Tr}\{P_{\hat{\theta}_{DML}} A_i A_i^\dagger (A[Z + E]_i)\}\right]
$$

$$
= -\frac{2}{m - d} \text{Re}\left[\hat{Z}_{ii} P_{\hat{\theta}_{DML}} A_i d_i + A_i^\dagger \hat{\Sigma} P_{\hat{\theta}_{DML}} d_i\right]
$$

(53)

where $\hat{Z}_{ii}$ is the $i$th row of $\hat{Z}$ and $\sigma = \partial a / \partial \theta_i$. Using (3) and (21), we get

$$
\text{E}[\hat{Z}_{ii} \hat{Z}_{ii}] = \frac{\sigma^2}{N} \hat{S}_{ii} I
$$

(54)

$$
\text{E}[\hat{Z}_{ii} \hat{Z}_{ii}] = 0
$$

(55)

$$
\text{E}[\hat{\Sigma}_{ij} \hat{\Sigma}_{ij}] = \frac{\sigma^4}{N} (\delta_{ij} - \delta_{i-k} - \delta_{-i-k} - \delta_{i-k})
$$

(56)

where $\hat{\Sigma}_{ij}$ denotes the $i$th component of $\hat{\Sigma}$. This leads, after some straightforward calculations, to

$$
\text{E}(\hat{\sigma}_i^2)^2 = \frac{2\sigma^2}{N(m - d)} \text{Re}\left[\text{Tr}\left\{P_{\hat{\theta}_{DML}} A_i A_i^\dagger \{\hat{S} + \sigma^2 (A^*A)^{-1})\}_{ii}\right\}\right]
$$

$$
\approx \frac{2\sigma^2}{N m^2} \text{Re}\left[\text{Tr}\left\{P_{\hat{\theta}_{DML}} A_i d_i\right\}\right].
$$

(57)

Inserting (57) into (50) shows that for large $m$

$$
\frac{\partial}{\partial \theta_i} f_m(\theta) \bigg|_{\theta = \theta_0} \approx O_p(|d_i|/m).
$$

(58)

Here, the notation $O_p(\cdot)$ and $o_p(\cdot)$ represent "in probability" counterparts of the corresponding deterministic notation (see Section 2.9 of [18]). The derivative of the first term of (38) is obtained as

$$
\frac{\partial}{\partial \theta_i} \text{log}|A^* \hat{R} A| = 2\text{Re}\left(\text{Tr}\{[A^* \hat{R} A]^{-1}(A^* \hat{R} A_i)\}\right).
$$

(59)

Assume that $\hat{S} > 0$. The dominant term of (59) for large $m$ evaluated at $\theta_0$ is then given by

$$
\text{Re}\left(\text{Tr}\{(A^* \hat{S} A A^* \hat{S} A)^{-1}(A^* \hat{S} A A^* \hat{S} A_i)\}\} = \text{Re}\left(\text{Tr}\{[A_i A_i^\dagger]\}\right).
$$

(60)

Thus

$$
\frac{\partial}{\partial \theta_i} \frac{1}{m - d} \text{log}|A^* \hat{R} A| \bigg|_{\theta = \theta_0}
$$

$$
\approx O_p\left(\frac{1}{m^2} \text{Tr}\{A_i A_i^\dagger\}\right) \approx O_p(|d_i|/m^{3/2}).
$$

(61)

Similarly, it follows for the second term in (38)

$$
\frac{\partial}{\partial \theta_i} \frac{1}{m - d} \text{log}|A^* A| \bigg|_{\theta = \theta_0}
$$

$$
\approx O_p\left(\frac{1}{m} \text{Tr}\{A_i A_i^\dagger\}\right) \approx O_p(|d_i|/m^{3/2}).
$$

(62)

Note that the derivative of the "noise" term (61) and (62) evaluated at $\theta_0$ is of the order $O_p(|d_i|/m^{3/2})$, whereas the "signal" term (58) is of order $O_p(|d_i|/m)$. Thus, the "noise" terms in (36) can be neglected for large $m$. We conclude that
the SML estimator is asymptotically given by
\[
\hat{\theta} = \arg\min_{\theta} \frac{1}{m-d} \text{Tr}(P_n^\perp(\theta) \hat{R})
\]
\[
= \arg\min_{\theta} \text{Tr}(P_n^\perp(\theta) \hat{R})
\]
which is recognized as the DML estimator. □

B. Beamforming

The beamforming technique requires asymptotically orthogonal array propagation vectors to yield consistent estimates. However, this is not quite enough to guarantee efficient parameter estimates. Since beamforming is identical to the DML method for \( d = 1 \), one expects the beamforming estimates to be efficient only when the CRB matrix is diagonal. By (62), this is true essentially only if the matrices (appropriately normalized)

\[
A^* A, \quad A^* D, \quad D^* D
\]
are all diagonal for large \( m \). We present a somewhat simplified proof of this claim in the following theorem.

**Theorem 6:** Assume that the array geometry is such that the matrices in (64) are asymptotically diagonal. Then, the beamforming estimates are asymptotically efficient.

**Proof:** The beamforming estimates are the local minima of the following criterion

\[
V(\theta) = \text{Tr}(P_n^\perp(\theta) \hat{R}).
\]

Since the estimates are consistent, a Taylor series expansion shows that the variance of \( \hat{\theta}_i \) is asymptotically given by

\[
E[(\hat{\theta}_i - \theta_i)^2] \approx \frac{E[V_i^2]}{(V_i)^2}
\]

where \( V_i \) and \( V_{ii} \) denote the first and second derivatives of \( V(\theta) \) evaluated at \( \theta_i \). Following (53)–(57), the first derivative is

\[
V_i(\theta) = 2 \text{Re} \{ \text{Tr}(P_n^\perp d_i d_i^\dagger \hat{R}) \}.
\]

By using the diagonality of \( A^* A \) and \( A^* D \), the dominant term of \( V_i \) is found to be

\[
V_i \approx 2 \text{Re} \{ Z_{ii} P_n^\perp d_i \}
\]

which gives

\[
E[V_i^2] \approx \frac{2\sigma^2}{N} \| Z_{ii} \|^2 P_n^\perp d_i d_i^\dagger S_{ii}.
\]

Similarly, the dominant term of \( V_{ii} \) is found to be

\[
V_{ii} \approx 2 d_i^\dagger P_n^\perp d_i S_{ii}.
\]

Inserting (69) and (70) into (66) gives

\[
E[(\hat{\theta}_i - \theta_i)^2] \approx \frac{\sigma^2}{2N d_i^\dagger P_n^\perp d_i S_{ii}} = \frac{1}{2N d_i^\dagger P_n^\perp d_i S N_i}.
\]

This is readily seen to coincide with the CRB (48) whenever the matrices in (64) are diagonal.

\[\Box\]

V. A NUMERICAL EXAMPLE

The performance analysis presented herein is unavoidably asymptotic in nature. It is, of course, of great interest to investigate the efficiency of the methods using arrays composed of a finite number of elements. Such a comparison is essentially possible only via computer simulations. Below, we present a simple example using a ULA with half a wavelength element spacing. Two completely coherent signals arrive from the DOA's 0° and 4°, respectively, and the element SNR is 0 dB for both signals. The correlation phase is 0° at the first sensor. The empirical results are computed from the sample statistics from a batch of 512 independent trials, each using \( N = 10 \) samples. Different noise realizations are generated in each trial, whereas the signal waveforms are fixed with sample covariance equal to

\[
\hat{S} = \begin{bmatrix}
0.98 & 0.98 \\
0.98 & 0.98
\end{bmatrix}.
\]

An iterative Newton-type technique is used for numerically minimizing the different optimization criteria. Since we are interested in the asymptotic behavior of the global optimum, the iterations are initialized at the true values. Fig. 2 displays the empirical RMS errors of \( \hat{\theta}_i \) for the various methods and the CRB versus the number of sensors. The results for \( \hat{\theta}_2 \) are virtually identical. The beamforming method fails to resolve the sources for \( m < 25 \), and the empirical RMS errors for those values are not included. Clearly, both ML methods achieve the CRB for very moderate values of \( m \). However, for \( m = 10 \), the DML method fails in 24% of the trials, whereas the SML technique has 18% failures. These failures are removed before computing the empirical RMS error. The standard deviation of the beamforming estimate is also very close to the CRB for \( m \geq 40 \). However, at \( m = 25 \) and \( m = 63 \), the RMS error is substantially larger than the CRB due to a large bias caused by the interference pattern of the two signals.

We also repeated the above experiment using uncorrelated signals. This gave very similar results, and this case is therefore omitted. From the above example, one may conjecture that

\[\Box\]

\[\Box\]

\[\Box\]

\[\Box\]

\[\Box\]

\[\Box\]
the ML methods are both “practically efficient” for small \( N \), provided \( m \) and SNR is large enough so that the theoretical standard deviation (the CRB) is smaller than, say, half the DOA separation. This is in contrast to the case where \( m \) and SNR are small whereas \( N \) is large. For this case, it is known that only the SML method gives efficient estimates, i.e., it attains the stochastic CRB, although the difference between SML and DML is notable only for highly correlated signals.

VI. CONCLUSIONS

The behavior of the deterministic and stochastic maximum likelihood methods as the number of sensors tends to infinity is studied. An identifiability condition is introduced, which essentially requires that the level of the beampattern at any fixed \( \theta \) must not approach the level of the main lobe for increasing \( m \). It is shown that both ML methods give consistent estimates of the DOA’s under this condition. The asymptotic efficiency of the DML method follows from the fact that all parameters are consistently estimated (unlike the case for large \( N \)). The same is not true for the stochastic ML method since the signal covariance matrix cannot be consistently estimated. However, the DML and SML estimates are found to be asymptotically equivalent, thus also implying asymptotic efficiency for the SML method.

It is shown that the traditional beamforming method gives consistent estimates if steering vectors corresponding to different DOA’s are asymptotically orthogonal. Since this technique coincides with the DML method for one signal, the beamforming estimates are asymptotically efficient if the CRB is asymptotically diagonal.

A Monte-Carlo simulation of a simple test case is included to study the asymptotic efficiency of the methods in arrays of finite size. Both ML estimators give efficient estimates using as few as \( m = 10 \) sensors in the studied scenario. Beamforming, on the other hand, requires a larger value of \( m \) for providing accurate estimates, due to the inherent problems with bias and resolution.

Interesting questions that deserve further study include a more precise classification of the set of array configurations that satisfy Assumption 1, as well as those for which the matrices in (64) (appropriately normalized) are diagonal. Note that these matrices are indeed diagonal for uniform linear arrays; see Appendix G of [5] for details. It is also of interest to investigate to what extent the assumptions on the noise made herein (complex Gaussian distribution, temporal, and spatial whiteness) can be relaxed.

It should be mentioned that our analysis can be extended to study the large \( m \) performance of other covariance-based array processing methods as well, using the techniques developed herein. The extension to subspace-based techniques is, however, not trivial. This is because the asymptotic behavior of the eigenvectors of \( \hat{R} \) is not known when its dimension \((m \times m)\), but not its sample size \((N)\), is increasing.

REFERENCES


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