Optimal Polarized Beampattern Synthesis Using a Vector Antenna Array

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Abstract—Using polarized waveforms increases the capacity of communication systems and improves the performance of active sensing systems such as radar. We consider the optimal synthesis of a directional beam with full polarization control using an array of electromagnetic vector antennas (EMVA). In such an array, each antenna consists of \( p \geq 2 \) orthogonal electric or magnetic dipole elements. The control of polarization and spatial power patterns is achieved through carefully designing the amplitudes and phases of the weights of these dipole antennas. We formulate the problem in a convex form, which is thus efficiently solvable by existing solvers such as the interior point method. Our results indicate that vector antenna arrays not only enable full polarization control of the beampattern, but also improve the power gain of the main beam (over the sidelobes), where the gain is shown numerically to be linearly proportional to the vector antenna dimensionality \( p \). This implies that EMVA not only offers the freedom to control the beampattern polarization, but also virtually increases the array size by exploiting the full electromagnetic (EM) field components. We also study the effect of polarization on the spatial power pattern. Our analysis shows that for arrays consisting of pairs of electrical and magnetic dipoles, the spatial power pattern is independent of the mainbeam polarization constraint.

Index Terms—Waveform polarization, vector antenna, beampattern synthesis, convex optimization

I. INTRODUCTION

It is well known that exploiting waveform polarization increases the capacity of communication systems and improves the performance of active sensing systems such as radar. In radar, polarimetric information in the backscattered waveforms contains the target features, such as geometrical structure, shape, reflectivity, and orientation, and can thus be exploited to significantly improve the sensing performance [1]–[9]. Capturing the backscattered polarimetric information requires the radar to measure both horizontal and vertical components of the received waveforms, which enables the so-called reception polarization diversity [3]. However, to efficiently acquire complete polarimetric information on the target, it is desirable for radar to be able to control the transmit waveform polarization, which in addition enables the transmit polarization diversity [6], [10], [11]. Among current radar systems, polarimetric radar that transmits waveforms with both horizontal and vertical orientations has been developed and adopted in a wide range of applications [12]. In fact, polarization diversity not only provides more complete information about the target, but also is a degree of freedom of the transmit waveforms that can be exploited in response to the dynamic targets and environments. Adaptive waveform polarization has been recently studied in the literature and has been shown to be an efficient approach to improve radar sensing performance in various scenarios [5], [13]–[15]. The focus of the above-mentioned work is on the analysis of performance improvement bought by adaptively controlling the transmit waveform polarization, but the issues regarding how to generate such polarized waveforms have not been addressed.

In wireless communications, waveform polarization has been shown to significantly improve the capacity of communication systems [16]. Polarization diversity in communications has also been well studied in the literature [17]–[22]. Classical wireless communications relies on one channel per frequency, although it is well understood that the two polarization states of planar waves allow two distinct information channels; techniques such as polarization diversity already take advantage of this. In [16] the authors further show that in a scattering environment, an extra factor of three in channel capacity can be obtained, relative to the limit imposed by using dual-polarized radio signals. The extra capacity arises because there are six distinguishable electric and magnetic states of polarization at a given point. In addition, polarized arrays can match the polarization of a desired signal and null an interferer with the same direction of arrival. The potential of polarized arrays for interference rejection in wireless communication systems has been investigated in recent years [23], [24].

In this paper, we study beampattern synthesis using an array of vector antennas. Each vector antenna in the array consists of \( p \geq 2 \) orthogonal electric or magnetic dipole elements. Our aim is to synthesize beampatterns with not only a desired spatial power pattern, but also a desired polarization. In traditional beampattern synthesis of a scalar array (i.e., an array of scalar antennas), radio signals from a set of small non-directional antenna elements are combined with different weights to achieve the beam spatial directivity. The beam emitted by such arrays is of a fixed polarization and cannot be controlled. To obtain a beam with full polarization control, we use an array of vector antennas. We design complex weights for individual array elements to achieve a beam with both the desired spatial power density and the desired
polarizations. Our goal is to explore the potential of EMVA from the transmitter point of view. By exploiting the full EM field components, we study how vector antennas can achieve polarization control and improve the spatial power pattern of the array.

There has been a long history of research on the problem of beampattern synthesis [25]-[28]. The main difference between the present paper and the previous work is that our work jointly designs the waveform polarization and the spatial power pattern, whereas work reported in the existing literature focuses on the latter only. We consider the joint design of the vector antenna array and compare its performance with the scalar arrays. We design the amplitudes and phases of the electric fields emitted from these dipole antennas to achieve the desired polarization control and spatial power pattern. By formulating the problem in an efficiently solvable convex optimization form, we can easily adopt various design criteria on the spatial power pattern and polarization by adding corresponding constraints. Our results indicate that our vector antenna not only allows full polarization control of the beampattern but also improves the power gain of the main beam (over the sidelobes), where the gain is linearly proportional to vector antenna dimensionality \( p \). This implies that EMVA has the advantage of not only providing the freedom to control the beampattern polarization but also virtually increasing the array size by exploiting full EM fields components. We also show that for arrays consisting of pairs of electrical and magnetic dipoles, the spatial power pattern is independent of the mainbeam polarization constraint.

Finally, we note that the performance of the minimum-noise-variance beamformer of a single EM vector antenna has been studied in [29]. The work therein considers the beamforming of the measurements of the complete electric and magnetic fields induced by EM signals from a single EM vector antenna. Our paper on the other hand, studies the optimal beamforming of an array of vector antennas.

**Organization of the paper:** The rest of the paper is organized as follows. In Section II, we give a brief overview of EM waveform polarization by introducing the polarization ellipse and a polarization transformation. Section III presents the problem formulation. In this section we first introduce the EMVA array and then give a convex formulation of the polarized waveform synthesis using an EMVA array. In Section IV, we study the cases for which each antenna in the array consists of a pair of electric and magnetic dipoles. We show that in such cases, the achievable beampattern spatial power gain is independent of the mainbeam polarization constraint. We also compare the vector array’s performance with its scalar counterpart. Section V is devoted to numerical simulations. Through numerical examples, we show that when \( p \geq 2 \), the spatial power gain (mainbeam over sidelobes) of the polarized beampattern is proportional to the antenna dimension. Section VI summarizes and concludes the paper.

**Notations:** Throughout this paper we adopt the following notations. A lower case letter (e.g., \( a \)) denotes a scalar, a boldface/lowercase letter (e.g., \( a \)) denotes a vector, and a boldface/uppercase letter (e.g., \( A \)) denotes a matrix. For a complex matrix \( A \in \mathbb{C}^{m \times n} \), \( A^T \) and \( A^H \) denote the transpose and Hermitian of \( A \), respectively. We use “\( j \)” to denote the complex symbol, i.e., \( j = \sqrt{-1} \). The letter \( I_n \) denotes an identity matrix of size \( n \times n \). In addition, all vectors are in column form unless otherwise noted.

**II. WAVEFORM POLARIZATION**

A plane wave (which is typically associated with a single source in the far field) radiated by an antenna comprises an electric field and a magnetic field that are orthogonal to the direction of propagation. The electric and magnetic fields are orthogonal to each other, and the magnitude of the electric and magnetic fields can be found from each other. Hence, only the electric field of the EM wave is usually used to describe its polarization.

**A. Polarization ellipse**

Consider a narrowband plane wave traveling in a uniform medium in the 3-D space. We assume that its electric field can be represented by

\[
\mathbf{E} = [E_H, E_V]^T = [\xi_H, \xi_V]^T s(t)
\]

where \( \xi_H \) and \( \xi_V \) are two complex numbers and \( s(t) \) is the scalar complex envelope of the waveform. The polarization is the locus of the electric field vector as a function of time; see Fig. 1. It describes the direction of wave oscillation in the plane perpendicular to the direction of propagation [30]. For ease of visualization, one efficient method of describing the waveform polarization is the so-called polarization ellipse\(^1\) [2], [4].

Given an electric field of a plane EM wave in (1), it turns out that the wave polarization status is fully determined by the vector \([E_H, E_V]^T\). The following theorem gives the relationship between the polarization status of \( \mathbf{E} \) and \([E_H, E_V]^T\).

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\( ^1\) Besides the polarization ellipse, there are other parametrizations for the waveform polarization, such as the Jones vector [31].
Theorem 1 ([2]): Every non-zero \( \mathbf{w} \in \mathbb{C}^2 \) has a unique representation:

\[
\mathbf{w} = ||\mathbf{w}|| e^{j\psi} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta \\ j \sin \beta \end{bmatrix}
\]

where \( \psi \in (-\pi, \pi) \), \( \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), and \( \beta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \). Moreover, \( \psi, \alpha, \beta \) in (2) are uniquely determined if and only if \( \xi_H^2 + \xi_V^2 \neq 0 \).

In (2), \( \alpha \) is called the orientation angle (the angle between the major semi-axis of the ellipse and the \( H \)-axis), and \( \beta \) is the ellipticity angle (the angle measuring the ratio of the two semi-axes); see Fig. 2. For example, when \( \beta = 0 \), the resultant polarization is linear; moreover, \( \alpha = 0 \) gives a horizontal polarization and \( \beta = \pi/2 \) leads to a vertical polarization. However, for \( \beta = \pm \pi/4 \), the resultant polarization is circular for any orientation angle \( \alpha \).

![Fig. 2. Polarization ellipse.](image)

In the next section, we will use the relationship between \( (\alpha, \beta) \) and \( (\gamma, \delta_\theta) \) to design the amplitudes and phases of each antenna sub-element to synthesize the polarized beampattern.

B. Polarization transformation

For convenience, we introduce the notation \( \mathcal{P}(\alpha, \beta) \) to denote the polarization status \( (\alpha, \beta) \). More specifically, for any pair of angles \( \alpha \in (-\pi/2, \pi/2), \beta \in [-\pi/4, \pi/4] \), we introduce the notation:

\[
\mathcal{P}(\alpha, \beta) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta \\ j \sin \beta \end{bmatrix} = R(\alpha) \begin{bmatrix} \cos \beta \\ j \sin \beta \end{bmatrix}
\]

where \( R(\alpha) \) denotes a rotation matrix of angle \( \alpha \). It turns out that any polarization \( \mathcal{P}(\alpha, \beta) \) can be mathematically transformed to \( \mathcal{P}(0, 0) \) by an orthogonal transformation. To show that, we introduce a linear transformation \( \mathbf{T}(\alpha, \beta) : [x_1, x_2]^T \rightarrow [y_1, y_2]^T \) in which

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{T}(\alpha, \beta) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos \beta & -j \sin \beta \\ j \sin \beta & -\cos \beta \end{bmatrix} R(-\alpha) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]

It is easy to verify that \( \mathbf{T}(\alpha, \beta) \) is an orthogonal transformation, and in addition,

\[
\mathbf{T}(\alpha, \beta) \mathcal{P}(\alpha, \beta) = \mathcal{P}(0, 0)
\]

Thus, for any pairs of polarizations \( \mathcal{P}(\alpha_1, \beta_1) \) and \( \mathcal{P}(\alpha_2, \beta_2) \), the transformation

\[
\mathbf{T}(\alpha_2, \beta_2; \alpha_1, \beta_1) \equiv \mathbf{T}^{-1}(\alpha_2, \beta_2) \mathbf{T}(\alpha_1, \beta_1)
\]

is orthogonal. Further, it maps \( \mathcal{P}(\alpha_1, \beta_1) \) to \( \mathcal{P}(\alpha_2, \beta_2) \), which is verified below:

\[
\begin{align*}
\mathbf{T}(\alpha_2, \beta_2; \alpha_1, \beta_1) \mathcal{P}(\alpha_1, \beta_1) &= \mathbf{T}^{-1}(\alpha_2, \beta_2) \mathbf{T}(\alpha_1, \beta_1) \\
&= \mathbf{T}^{-1}(\alpha_2, \beta_2) \mathcal{P}(0, 0) \\
&= \mathcal{P}(\alpha_2, \beta_2).
\end{align*}
\]

We will use such transformations in Section IV and V to simplify the design of a polarized beampattern and show the property of polarized beampattern synthesis.

III. PROBLEM FORMULATION

In this section we formulate the problem of the optimal beampattern synthesis with full polarization control. We first introduce the configurations of the EMVA and the EMVA array used in this paper, and then formulate the beampattern synthesis problem using tools from convex optimization.

A. Vector antenna response

![Fig. 3. The coordinate system \((r, r_H, r_V)\).](image)

Let \( \mathbf{r} \) be a unit vector representing a spatial direction in \( \mathbb{R}^3 \). We can write \( \mathbf{r} \) in the form

\[
\mathbf{r} = \begin{bmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ \sin \theta \end{bmatrix}
\]
where $-\pi/2 \leq \phi \leq \pi/2$ is the azimuth angle and $0 \leq \theta < 2\pi$ is the elevation angle. For each $r$, we further choose

$$
\mathbf{r}_H \overset{\text{def}}{=} \frac{1}{\cos \phi} \frac{\partial \mathbf{r}}{\partial \phi} = \begin{bmatrix}
-sin \phi \\
\cos \phi \\
0
\end{bmatrix} \\
\mathbf{r}_V \overset{\text{def}}{=} \frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix}
-cos \phi \sin \theta \\
-sin \phi \sin \theta \\
cos \theta
\end{bmatrix}
$$

It is easy to see that $(\mathbf{r}, \mathbf{r}_H, \mathbf{r}_V)$ forms a right-hand coordinate system; see Fig. 3. For a plane wave traveling along $\mathbf{r}$, the electric field is orthogonal to $\mathbf{r}$ and lies in the plane spanned by $(\mathbf{r}_H, \mathbf{r}_V)$.

Above $\mathbf{v}_x^{(E)}, \mathbf{v}_y^{(E)},$ and $\mathbf{v}_z^{(E)}$ denote the response of the three electric dipoles located at the $x, y,$ and $z$-axis respectively. Similar definitions follow for $\mathbf{v}_x^{(M)}, \mathbf{v}_y^{(M)},$ and $\mathbf{v}_z^{(M)}$.

It is readily seen that there is a reciprocal property between the electric dipole and magnetic dipole along the same axis. Specifically, we have

$$
\begin{bmatrix}
\mathbf{v}_x^{(M)} \\
\mathbf{v}_y^{(M)} \\
\mathbf{v}_z^{(M)}
\end{bmatrix} = \begin{bmatrix}
\mathbf{v}_x^{(E)} \\
\mathbf{v}_y^{(E)} \\
\mathbf{v}_z^{(E)}
\end{bmatrix} \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
$$

This is a key property that will be used in Section IV to discuss the relationship between the beam pattern spatial power gain and the beam polarization constraint.

For a vector antenna consisting of $p$ dipole elements, we use $\mathbf{V}(\mathbf{r})$ to denote the antenna response. In this paper, we will consider antenna types with $2 \leq p \leq 6$. The scalar antenna array case with $p = 1$ will also be considered for comparison purposes. For example, if an antenna consists of one electric dipole element at the $x$-axis only, its response $\mathbf{V}(\mathbf{r}) = \mathbf{v}_x^{(E)}(\mathbf{r})$. If the antenna consists of both electric and magnetic dipole elements along the $x$ axis, the response $\mathbf{V}(\mathbf{r}) = \begin{bmatrix} \mathbf{v}_x^{(E)} \\ \mathbf{v}_x^{(M)} \end{bmatrix}$.

Moreover, for a given antenna response $\mathbf{V}(\mathbf{r}) \in \mathbb{C}^{p \times 2}$, we use $\mathbf{v}(\mathbf{r}; H)$ and $\mathbf{v}(\mathbf{r}; V)$ to denote its response to the $H$ and $V$ dimensions, respectively, or as a formula,

$$
\mathbf{V}(\mathbf{r}) = \begin{bmatrix} \mathbf{v}(\mathbf{r}; H) \\ \mathbf{v}(\mathbf{r}; V) \end{bmatrix}.
$$

Equivalently, $\mathbf{v}(\mathbf{r}; H)$ and $\mathbf{v}(\mathbf{r}; V)$ are the two columns of $\mathbf{V}(\mathbf{r})$.

**B. Vector antenna array**

The antenna array we consider in this paper consists of $N$ EMVs located at $N$ positions in $\mathbb{R}^3$ with coordinates $\mathbf{x}_n : 1 \leq n \leq N$. Each antenna in the array has $p$ dipole elements. The antennas are driven by the same carrier signal with wavelength $\lambda$ and convex envelope $s(t)$. The antenna currents (or weights) on the $p$ dipoles in the $n$-th EMVA are denoted by (see Fig. 6)

$$
\mathbf{w}_n = [w^{(1)}_n, w^{(2)}_n, \ldots, w^{(p)}_n]^T; \quad 1 \leq n \leq N
$$

We further introduce $\mathbf{w}$ to be the concatenation of all $\mathbf{w}_n$:

$$
\mathbf{w} = [\mathbf{w}_1^T, \mathbf{w}_2^T, \ldots, \mathbf{w}_N^T]^T.
$$

It is easy to see that the dimensionality of $\mathbf{w}$ is $pN \times 1$.

Given location $\mathbf{x}_n : 1 \leq n \leq N$ of the $N$ antennas, the array response, as a function of spatial direction $\mathbf{r}$, can be expressed as

$$
\mathbf{a}(\mathbf{r}) = [e^{-j \psi_1(\mathbf{r})}, e^{-j \psi_2(\mathbf{r})}, \ldots, e^{-j \psi_N(\mathbf{r})}]^T,
$$

where $\psi_n(\mathbf{r}) = k \cdot \mathbf{x}_n$ and $k = 2\pi/\lambda$ is the wave number.

Thus, in terms of the vector antenna response $\mathbf{V}(\mathbf{r})$ (which has dimension $p \times 2$), we further obtain that the vector antenna
array response is

\[
A(r) = a(r) \otimes V(r) = \begin{bmatrix}
e^{-j\psi_1(r)}V(r) \\
e^{-j\psi_2(r)}V(r) \\
\vdots \\
e^{-j\psi_N(r)}V(r)
\end{bmatrix}
\]

which has dimensionality \( pN \times 2 \).

We introduce the antenna array response along the \( H \) and \( V \) directions:

\[
A(r; H) = a(r) \otimes v(r; H),
A(r; V) = a(r) \otimes v(r; V)
\]

where \( v(r; H) \) and \( v(r; V) \) are defined as in (10).

![Vector antenna array](image)

The normalized electrical field emitted from the antenna array (ignoring the common carrier and the baseband signal \( s(t) \)) can be expressed as

\[
E(r) = A(r)^T w.
\]

Let us use \( E(r; H) \) and \( E(r; V) \) to denote the decomposition of \( E(r) \) along \( H \) and \( V \). We immediately have

\[
E(r; H) = A(r; H)^T w,
E(r; V) = A(r; V)^T w.
\]

Along \( r \), the wave polarization is determined by the ratio between \( E(r; V) \) and \( E(r; H) \), and the radiated energy can be expressed as \( ||E(r)||^2 = |E(r; H)|^2 + |E(r; V)|^2 \).

\[\text{C. Polarized beampattern synthesis}\]

The problem of beampattern synthesis is to design the antenna weights \( w \) to achieve a desired wave pattern. In a scalar array, the goal is merely to control the spatial power pattern. However, for the array of vector antennas, we can further achieve the control of beam polarization. Specifically, our goal is to synthesize a beampattern with the following properties:

i) The mainbeam, assumed to be pointing at a direction \( r_0 \), has desired power \( P \) and polarization \( (\alpha, \beta) \);

ii) The power of sidelobes in a region of interest (denoted by \( S_p \)) are suppressed.

Compared with the scalar array, the beampattern synthesis in a vector array enforces an additional polarization constraint on the main beam.

There are various criteria for suppressing the sidelobe power while maintaining the mainlobe power and polarization. We focus on the sidelobe power minimization in this paper, and this leads to the following optimization problem:

\[
\begin{align*}
\min_{w} & \quad \max_{r_x \in S_r} ||E(r_x)||^2 \\
\text{s.t.} & \quad E(r_0) = \sqrt{P} e^{j\theta} \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix} \begin{bmatrix}
\cos \beta \\
j \sin \beta
\end{bmatrix}, \\
E(r) &= A(r)^T w, \quad \forall r
\end{align*}
\]

To obtain the optimal weights \( w \) for the \( Np \) antenna elements, we need to solve the above optimization problem. To remove the “max” operation in the objective function and transform it into a standard function, we introduce an auxiliary variable \( \tau \). We further assume \( \sqrt{P} e^{j\theta} = 1 \) without loss of generality. Then we obtain the following problem:

\[
\begin{align*}
\min_{w, \tau} & \quad \tau \\
\text{s.t.} & \quad ||E(r_x)||^2 \leq \tau; \quad \forall r_x \in S_r \\
E(r_0) &= \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix} \begin{bmatrix}
\cos \beta \\
j \sin \beta
\end{bmatrix}, \\
E(r) &= A(r)^T w, \quad \forall r
\end{align*}
\]

It is easy to see that the above problem is convex, and is in fact a second order cone programming (SOCP) problem [32].

SOCP is a special class of convex optimization problems and therefore enjoys all the advantages of convexity. There are well-developed numerical methods to solve a general convex optimization problem, the best known of which is the interior point method. For the numerical examples in Section V, we adopt an optimization toolbox, SeDuMi [33], to solve the SOCP formulated above. SeDuMi, which stands for Self-Dual-Minimization, is a software package that solves optimization problems over symmetric cones using the primal-dual interior-point methods.

In the next two sections, we discuss the polarized beampattern synthesis and analyze its performance for two cases, \( p = 2 \) and \( p \geq 3 \), where \( p \) is the number of dipole elements in each vector antenna. Specifically, for \( p = 2 \), we will compare its performance with its scalar array counterpart, and for \( p \geq 3 \), we will analyze the mainbeam power gain versus \( p \).

\[\text{IV. DISCUSSION AND PERFORMANCE ANALYSIS WHEN} \quad p = 2\]

In this section, we study the beampattern synthesis using 2-D vector antenna arrays, where each vector antenna has one electric dipole and one magnetic dipole co-aligned along either the \( x \), \( y \), or \( z \) axis. Our aims are twofold: (i) to find the relationship between beampattern power gain and its
polarization constraint; and (ii) to compare the performance of such 2-D vector arrays with their scalar counterparts.

Depending on which pair of dipoles is chosen, there are three possibilities for the vector antenna response matrix \( V(r) \). In each case, \( V(r) \) has the following structure:

\[
V(r) = \begin{bmatrix}
u(r) & v(r) \\
v(r) & -u(r)
\end{bmatrix}
\]

where

- For dipoles along the \( x \)-axis, \([u(r), v(r)] = v_x^{(E)}(r) = [-\sin \phi, -\cos \phi \sin \theta];\)
- For dipoles along the \( y \)-axis, \([u(r), v(r)] = v_y^{(E)}(r) = [\cos \phi, -\sin \phi \sin \theta];\) and
- For dipoles along the \( z \)-axis, \([u(r), v(r)] = v_z^{(E)}(r) = [0, \cos \theta].\)

For all these cases, it holds that

\[
V(r)^T V(r) = (|u(r)|^2 + |v(r)|^2) I_2.
\] (13)

![Fig. 7. Array of 2-D paired dipoles.](image)

Following the notations introduced in (11), the weights at the two dipole elements for the \( i \)-th vector antenna are \( w_{i}^{(1)} \) and \( w_{i}^{(2)} \). For convenience, we use \( w_{i}^{(E)} \equiv w_{i}^{(1)} \) in this section to denote the weight on the electric dipole, and \( w_{i}^{(M)} \equiv w_{i}^{(2)} \) to denote the weight on the magnetic dipole; see Fig. 7. We further introduce the following notations:

\[
w^{(E)} \equiv \begin{bmatrix} w_1^{(E)} \\ w_2^{(E)} \\ \vdots \\ w_N^{(E)} \end{bmatrix}, \quad w^{(M)} \equiv \begin{bmatrix} w_1^{(M)} \\ w_2^{(M)} \\ \vdots \\ w_N^{(M)} \end{bmatrix}.
\]

Notice that \( w^{(E)} \) and \( w^{(M)} \) are row vectors.

For this 2-D case, we further calculate the electrical field emitted from the antenna array below:

\[
E(r) = A(r)^T w
= (a(r) \otimes V(r))^T w
= \sum_{n=1}^{N} a_n(r) V(r)^T \begin{bmatrix} w_n^{(E)} \\ w_n^{(M)} \end{bmatrix}
= \sum_{n=1}^{N} a_n(r) \begin{bmatrix} w_n^{(E)} \\ w_n^{(M)} \end{bmatrix}
= \begin{bmatrix} s^{(E)}(r) \\ s^{(M)}(r) \end{bmatrix}
\] (14)

where

\[
s^{(E)}(r) \equiv w^{(E)} a(r), \quad s^{(M)}(r) \equiv w^{(M)} a(r).
\]

Using (13), we obtain

\[
||E(r)||^2 = (|u(r)|^2 + |v(r)|^2) (|s^{(E)}(r)|^2 + |s^{(M)}(r)|^2).
\]

In terms of \( s^{(E)}(r) \) and \( s^{(M)}(r) \), the original optimization problem in (12) becomes

\[
\min_{w: \tau} \tau \\
\text{s.t.} \\
\quad ||E(r_0; H)|| \leq \cos \alpha - \sin \alpha \quad \text{and} \\
\quad ||E(r_0; V)|| \leq \sin \alpha \cos \alpha \\
\quad ||E(r_0; H)||^2 + ||E(r_0; V)||^2 \leq \tau^2; \quad \nu \in S_r \\
\quad E(r; H) = \begin{bmatrix} u(r) \end{bmatrix}, \quad E(r; V) = \begin{bmatrix} v(r) \end{bmatrix} \\
\quad s^{(E)}(r) = w^{(E)} a(r), \quad s^{(M)}(r) = w^{(M)} a(r)
\] (16)

We will show that for the above problem, the following results hold:

i) The optimal solution \( \tau^{\text{opt}} \), which measures the beampattern power gain over the sidelobes, does not depend on the main beam polarization \( (\alpha, \beta) \); and
ii) The beampattern spatial power gain is the same when compared to its scalar array counterpart in which each sensor only consists of only one electrical (or magnetic) dipole. However, the latter scalar array does not have the ability to control the beampattern polarization.

A. Proof of independence of the polarization \( \mathcal{P}(\alpha, \beta) \)

For any arbitrary pair \( (\alpha, \beta) \), we assume the optimal solution to (16) is denoted by \( \tau^{\text{opt}}(\alpha, \beta) \). We will show that the following lemma holds.

Lemma 2: Assume that \( \tau^{\text{opt}}(\alpha, \beta) \) is the optimal solution to (16). For any \( (\alpha_1, \beta_1) \) and \( (\alpha_2, \beta_2) \), it holds that

\[
\tau^{\text{opt}}(\alpha_1, \beta_1) = \tau^{\text{opt}}(\alpha_2, \beta_2).
\]

Proof: Suppose \( \mathcal{P}(\alpha_1, \beta_1) = \mathcal{P}(\alpha_2, \beta_2) \) are two arbitrary waveform polarizations; see (3). As verified in Section II-B, if we introduce an orthogonal transformation \( T(\alpha_2, \beta_2; \alpha_1, \beta_1) \) (c.f., (5)), it holds that

\[
\mathcal{T}(\alpha_2, \beta_2; \alpha_1, \beta_1) \mathcal{P}(\alpha_1, \beta_1) = \mathcal{P}(\alpha_2, \beta_2).
\]

For a given direction \( r_0 \), we further define the transformation

\[
\mathcal{T}(r_0; \alpha_2, \beta_2; \alpha_1, \beta_1) = \mathcal{V}(r_0)^{-1} \mathcal{T}(\alpha_2, \beta_2; \alpha_1, \beta_1) \mathcal{V}(r_0)
\]

where

\[
\mathcal{V}(r_0) = \begin{bmatrix} u(r_0) & v(r_0) \\ v(r_0) & -u(r_0) \end{bmatrix}
\]

It is easy to see that \( \mathcal{T}(r_0; \alpha_2, \beta_2; \alpha_1, \beta_1) \) is an orthogonal transformation.
Suppose \( \begin{bmatrix} w_1^{(E)opt} \\ w_1^{(M)opt} \end{bmatrix} \) is an optimal solution to problem (16) when the mainbeam points at direction \( r_0 \) and has polarization constraint \((\alpha_1, \beta_1)\). The corresponding optimal objective value is denoted by \( \tau^{opt}(\alpha_1, \beta_1) \). Recalling the first and third constraints in (16), we obtain that

\[
\begin{bmatrix} E(r_0; H) \\ E(r_0; V) \end{bmatrix} = V(r_0) \begin{bmatrix} s_1^{(E)opt} \\ s_1^{(M)opt} \end{bmatrix} = \mathcal{P}(\alpha_1, \beta_1). \tag{17}
\]

Consider problem (16) with another polarization constraint \((\alpha_2, \beta_2)\). We choose the following antenna weights

\[
\begin{bmatrix} w_2^{(E)*} \\ w_2^{(M)*} \end{bmatrix} = T(r_0; \alpha_2, \beta_2; \alpha_1, \beta_1) \begin{bmatrix} w_1^{(E)opt} \\ w_1^{(M)opt} \end{bmatrix}. \tag{18}
\]

We first claim that the above chosen \( \begin{bmatrix} w_2^{(E)*} \\ w_2^{(M)*} \end{bmatrix} \) is a feasible solution to problem (16) with mainbeam polarization constraint \((\alpha_2, \beta_2)\). For that purpose, we verify that

\[
\begin{bmatrix} s_2^{(E)}(r)^* \\ s_2^{(M)}(r)^* \end{bmatrix} = \begin{bmatrix} w_2^{(E)*} \\ w_2^{(M)*} \end{bmatrix} a(r)
\]

where the first equality is due to (15).

Therefore, at \( r_0 \), the resultant electrical field

\[
\begin{bmatrix} E_2^{(H)}(r_0)^* \\ E_2^{(V)}(r_0)^* \end{bmatrix} = V(r_0) \begin{bmatrix} s_2^{(E)}(r_0)^* \\ s_2^{(M)}(r_0)^* \end{bmatrix}
\]

\[
= V(r_0) T(r_0; \alpha_2, \beta_2; \alpha_1, \beta_1) \begin{bmatrix} s_1^{(E)opt} \\ s_1^{(M)opt} \end{bmatrix}
\]

\[
= V(r_0) V(r_0)^{-1} T(\alpha_2, \beta_2; \alpha_1, \beta_1) V(r_0) \begin{bmatrix} s_1^{(E)opt} \\ s_1^{(M)opt} \end{bmatrix}
\]

\[
= T(\alpha_2, \beta_2; \alpha_1, \beta_1) \mathcal{P}(\alpha_1, \beta_1)
\]

where (a) is due to (18), (b) is because of the definition of \( T(r_0; \alpha_2, \beta_2; \alpha_1, \beta_1) \), and (c) is from (17). Further, we can derive from (19) that

\[
\left| s_2^{(E)}(r)^{opt} \right|^2 + \left| s_2^{(M)}(r)^{opt} \right|^2 = \left| s_1^{(E)}(r)^{opt} \right|^2 + \left| s_1^{(M)}(r)^{opt} \right|^2
\]

for any \( r \).

We conclude from (20) and (21) that \( \begin{bmatrix} w_2^{(E)*} \\ w_2^{(M)*} \end{bmatrix} \) is a solution to problem (16) when the mainbeam polarization constraint is \((\alpha_2, \beta_2)\). Moreover, the corresponding objective value is \( \tau^{opt}(\alpha_2, \beta_2) \). This implies that

\[
\tau^{opt}(\alpha_2, \beta_2) \leq \tau^{opt}(\alpha_1, \beta_1).
\]

We can similarly prove that \( \tau^{opt}(\alpha_1, \beta_1) \leq \tau^{opt}(\alpha_2, \beta_2) \).

\[
\tau^{opt}(\alpha_1, \beta_1) = \tau^{opt}(\alpha_2, \beta_2). \quad \Box
\]

Essentially Lemma 2 reveals the fact that the mainbeam power gain is independent of the polarization constraint when each vector sensor in the array consists of a pair of co-aligned electric and magnetic dipoles.

### B. Comparison with scalar antenna arrays

In this section, we compare the performance of the 2-D antenna array (as displayed in Fig. 7) with its scalar counterpart. In the corresponding scalar array, each antenna consists of either an electric or a magnetic dipole; see Fig. 8. Such scalar arrays do not allow mainbeam polarization control due to the lack of degree of freedom. In fact, for any spatial direction \( r \), the polarizations of the wave radiated by all antennas in the scalar array are identical. Thus, the resultant wave radiated by the array remains fixed no matter how the weights for the antenna elements are designed. We hence discard the polarization constraint for the problem of beampattern synthesis using such scalar arrays and consider only the spatial power distribution.

![Fig. 8. Scalar arrays: (a) array of electric dipoles; (b) array of magnetic dipoles](image)

The following two problems formulate the beampattern synthesis using the array of electric dipoles and the array of magnetic dipoles, respectively:

- **1-D array with electric dipoles:** The antenna response \( V(r) = [u(r), v(r)] \) and the optimal antenna weights \( w^{(E)} \) can be obtained by solving

\[
\min_{w^{(E)}} \quad \tau
\]

\[
s.t. \quad ||E(r_0; H)||^2 + ||E(r_0; V)||^2 = 1
\]

\[
||E(r_s; H)||^2 + ||E(r_s; V)||^2 \leq \tau^2; \quad r_s \in \mathcal{S}_r
\]

\[
E(r; H) = \begin{bmatrix} u(r) \\ v(r) \end{bmatrix} s^{(E)}(r)
\]

\[
\tau = |s^{(E)}(r)| = w^{(E)a}(r)
\]

- **1-D array with magnetic dipoles:** The antenna response \( V(r) = [v(r), -u(r)] \) and the optimal antenna weights \( w^{(M)} \) can be obtained by solving

\[
\min_{w^{(M)}} \quad \tau
\]

\[
s.t. \quad ||E(r_0; H)||^2 + ||E(r_0; V)||^2 = 1
\]

\[
||E(r_s; H)||^2 + ||E(r_s; V)||^2 \leq \tau^2; \quad r_s \in \mathcal{S}_r
\]

\[
E(r; H) = \begin{bmatrix} v(r) \\ -u(r) \end{bmatrix} s^{(M)}(r)
\]

\[
\tau = |s^{(M)}(r)| = w^{(M)a}(r)
\]
It is easy to see that the above two problems have the same optimal solution because

\[ \|E(r; H)\|^2 + \|E(r; V)\|^2 = (|u(r)|^2 + |v(r)|^2) |s^{(E)}(r)|^2 \]
\[ \|E(r; H)\|^2 + \|E(r; V)\|^2 = (|u(r)|^2 + |v(r)|^2) |s^{(M)}(r)|^2 \]

holds for (21) and (22) respectively, and, in addition, the two arrays share the same array response \( a(r) \).

For given array response \( a(r) \), antenna response \( (u(r), v(r)) \), and mainbeam direction \( \mathbf{r}_0 \), we use \( \tau_{\text{opt}} \) to denote the value of the objective function at the solutions for (21) and (22). We further have the following lemma, which states the relationship of the power gain achieved by these two scalar arrays and the vector array in (16).

**Lemma 3:** For any \( (\alpha, \beta) \), it holds that

\[ \tau_{\text{opt}} = \tau_{\text{opt}}(\alpha, \beta). \]

**Proof:** See Appendix.

The results given in Lemmas 2 and 3 will be illustrated in the next section by numerical examples. We note that the results in Lemma 2 and 3 hold for the case of \( p = 2 \) only. Similar results for \( p \geq 3 \) are still under investigation.

V. NUMERICAL RESULTS

In this section we give numerical examples to show the performance of the proposed beampattern synthesis method. We will consider two cases: \( p = 2 \) and \( p \geq 3 \). Since all formulated problems in (12), (16), (21), and (22) are convex, we adopt the optimization toolbox, SeDuMi [33] to solve the formulated problems.

A. 2-D arrays: vector vs. scalar

The analysis in Section IV implies that for the 2-D vector arrays with co-aligned electric and magnetic dipoles, the achievable spatial power pattern is identical to that achieved by the scalar arrays consisting of either electric or magnetic dipoles. However, the vector array has the advantage of enabling the control of the polarization. In addition, as revealed by (25) in the Appendix, designing the weights for the vector array can be achieved by designing the two subarrays separately and then combining these weights properly without loss of optimality.

Fig. 9 shows the computer simulation of a linear array of 15 elements that are located on the \( z \)-axis and separated by a half wavelength, i.e.,

\[ \{x_n : 1 \leq n \leq 15\} = \left\{ 0, 0, (m - 9.5) \frac{\lambda}{2} \right\}^T : 1 \leq m \leq 18 \}

In the vector array, each antenna consists of one electric dipole and one magnetic dipole, both pointing along the \( z \)-axis. The
scalar array compromises 18 electric dipoles. The main beam is targeted at \( \theta_0 = 10^\circ \) with a beamwidth 7.5°. The main beam of the vector array has a polarization constraint \((\alpha, \beta) = (40^\circ, 10^\circ)\). As can be seen, both scalar and vector arrays achieve the same spatial power pattern, whereas the beam from the vector array has the desired polarization property.

Another example is given in Fig. 10. There are 18 antennas in each array. The antennas are located at the three axes with coordinates:

\[
\{ x_n : 1 \leq n \leq 18 \} = \begin{cases} 
( m - 3.5 \frac{\lambda}{2} , 0 , 0 )^T : 1 \leq m \leq 6 \\
( 0 , ( m - 3.5 ) \frac{\lambda}{2} , 0 )^T : 1 \leq m \leq 6 \\
( 0 , 0 , ( m - 3.5 ) \frac{\lambda}{2} )^T : 1 \leq m \leq 6 
\end{cases}
\]

(23)

where \( \lambda \) is the wavelength. Still in the vector array, each antenna consists of one electric dipole and one magnetic dipole, both pointing along the \( z \)-axis. The two scalar arrays compromise electric dipoles only or magnetic dipoles only. The main beam points at direction \( r_0 = [\theta(r_0), \phi(r_0)] = [45^\circ, 45^\circ] \) with beamwidth 5° for both the elevation and azimuth angles. As can be seen, the three arrays achieve the same spatial power pattern, but the vector array also achieves a mainbeam control where, in this example, the polarization of the mainbeam is set to be \((\alpha, \beta) = [-30^\circ, 45^\circ]\).

B. High-dimensional arrays: power gain vs. sensor dimensionality

In this section we analyze by simulation the performance of beam pattern synthesis using arrays of high-dimensional antenna arrays. We still use array size 18 with antennas located at the \( x, y \), and \( z \) axis as given in (23). We will consider arrays of vectors antennas with \( 2 \leq p \leq 6 \). For each \( p \), we choose dipole elements according to Table I. We note that at each row, the symbol \( \sqrt{\ } \) denotes that a certain dipole element is included in the array. For example, when \( p = 2 \), one electric dipole and one magnetic dipole at the \( z \)-axis are selected. Correspondingly, the antenna response

\[
V(r) = \begin{bmatrix}
V_z^{(E)}(r) & V_z^{(M)}(r)
\end{bmatrix}^T
\]

(c.f., (7)–(9)).

<table>
<thead>
<tr>
<th>Table I</th>
<th>Vector antenna dipole elements for ( 2 \leq p \leq 6 ) that are used in the simulations in Section VI.B.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 2 )</td>
<td>( V_x^{(E)} ) √ ( V_y^{(E)} ) √ ( V_z^{(M)} ) √ √ √</td>
</tr>
<tr>
<td>( p = 3 )</td>
<td>√ √ √ √ √ √ √</td>
</tr>
<tr>
<td>( p = 4 )</td>
<td>√ √ √ √ √ √ √</td>
</tr>
<tr>
<td>( p = 5 )</td>
<td>√ √ √ √ √ √ √</td>
</tr>
<tr>
<td>( p = 6 )</td>
<td>√ √ √ √ √ √ √</td>
</tr>
</tbody>
</table>

In addition, in these examples, we choose \( r_0 = [\theta(r_0), \phi(r_0)] = [45^\circ, 45^\circ] \). The polarization of the mainbeam is set to be \((\alpha, \beta) = [30^\circ, 45^\circ]\).

The achieved spatial power patterns for \( p = 3, 4, 6 \) are given in Fig. 11–Fig. 13, respectively. For each case, the top figure plots the two-dimensional power pattern in the plane of the elevation and azimuth angles \((\theta, \phi)\). In the bottom figure, each curve shows the slice of the power pattern versus \( \theta \) for fixed \( \phi \). As can be seen, once the sensor dimensionality \( p \) increases, the mainbeam power gain versus the sidelobes increases. As plotted in Fig. 14, when \( p \) increases from 1 to 2, polarization control is enabled while the power gain remains the same (for \( p = 1 \), we choose the array of electric dipoles pointing along the \( z \)-axis). For the vector sensor arrays (when \( p \geq 2 \)), all three curves for different array size \( N = 6, 12, 18 \) show that such gain is almost linearly proportional to \( p \). We thus conclude that the EMVA array has the advantage of

i) enabling control of the beam pattern polarization; and

ii) virtually increasing the array size, since multiple EM fields at each antenna are exploited.

In other words, by exploiting the electrical and magnetic components radiated at the same physical location by a vector antenna, we can not only obtain the full control of the waveform polarization, but also improve the array resolution. The resolution increase of the EMVA array is due to the almost
independent responses from antenna elements from the same location. In applications where antenna array physical size is limited by space, the result above shows that the array resolution can go beyond the limit imposed by a scalar array if on top of the space dimension, the polarization dimension is exploited.

Fig. 12. Illustration of mainbeam power gain: \( p = 4 \).

VI. CONCLUSION

In this paper we have considered the beampattern synthesis with polarization constraints using an array of vector antennas consisting of electric and magnetic dipole elements. We formulated the problem in a convex form, for which the design variables are the amplitudes and phases of the weights in the antenna elements, and the polarization condition is cast in a linear constraint. We have shown that arrays consisting of 2-D vector antennas, where each antenna compromises a pair of co-aligned electric and magnetic dipoles, have the same capability of suppressing the sidelobe power density as the corresponding scalar arrays of electric or magnetic dipoles. However, the vector array has the additional capability of controlling the beam polarization. For arrays of vector antennas with the number of dipole elements in each antenna \( p \geq 2 \), we have shown by simulation that the power gain achieved by the array is linearly proportional to \( p \). This reveals that vector antenna arrays not only enable polarization control, but also virtually increase the array size by exploiting multiple EM fields at each physical point.

Fig. 13. Illustration of mainbeam power gain: \( p = 6 \).

Fig. 14. Power gain versus antenna dimensionality \( p \) (note that polarization control is enabled if and only if \( p \geq 2 \)).
Note that although we have considered only the transmit beampattern synthesis, similar results can be generalized to the receive beamforming of vector antenna arrays. Also the beampattern synthesis problem in this paper is formulated based upon the assumption of a narrowband plane wave traveling in a uniform medium. Its extension to the wideband waveforms in inhomogeneous medium is part of our future work.

APPENDIX
PROOF OF Lemma 3

As we demonstrated in Section IV-B, (21) and (22) share the same set of optimal solutions. Take one of them, which we denote by $w^{(E)}_{opt}$ and $w^{(M)}_{opt}$, respectively. We use $\tau^{opt}$ to denote the achieved optimal objective value. This implies that for any $r \in S_r$

$$s^{(E)}(r) = w^{(E)}_{opt} a(r),$$
$$s^{(M)}(r) = w^{(M)}_{opt} a(r),$$

it holds that

$$\left( |u(r)|^2 + |v(r)|^2 \right) |s^{(E)}(r)|^2 \leq \tau^{opt},$$
$$\left( |u(r)|^2 + |v(r)|^2 \right) |s^{(M)}(r)|^2 \leq \tau^{opt}.$$ (24)

In addition

$$\left( |u(r_0)|^2 + |v(r_0)|^2 \right) |s^{(E)}(r_0)|^2 = 1$$
$$\left( |u(r_0)|^2 + |v(r_0)|^2 \right) |s^{(M)}(r_0)|^2 = 1.$$ (25)

Let us construct a $w(U)$ (independent of $r$) such that

$$w = U \begin{bmatrix} w^{(E)}_{opt} \\ w^{(M)}_{opt} \end{bmatrix}$$

where

$$UU^H = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

with $\lambda_1 + \lambda_2 = 1$. Such a $w$ is a solution to (16) which gives an objective value $\tau^{opt}$ for those $U$ further satisfying the following:

$$\begin{bmatrix} u(r_0) & v(r_0) \\ v(r_0) & -u(r_0) \end{bmatrix} \begin{bmatrix} s^{(E)}(r_0) \\ s^{(M)}(r_0) \end{bmatrix} = \mathcal{P}(\alpha, \beta).$$

(27)

Obviously, there are infinitely many such $U$. For example, we can choose a diagonal

$$U = \begin{bmatrix} \frac{v_1}{s^{(E)}(r_0)} & 0 \\ 0 & \frac{v_2}{s^{(M)}(r_0)} \end{bmatrix}$$

in which

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u(r_0) & v(r_0) \\ v(r_0) & -u(r_0) \end{bmatrix}^{-1} \mathcal{P}(\alpha, \beta).$$

(28)

It is easy to see that $U$ is independent of $r$, $U$ satisfies (27), and $U$ further satisfies (26) because of (24). Therefore, the $w$
given in (25) is a solution to (16), which gives an objective value $\tau^{opt}$. This implies that

$$\tau^{opt} \geq \tau^{opt}(\alpha, \beta).$$ (29)

On the other hand, for problem (16), we choose the polarization constraint at $r_0$ to be $(\alpha(r_0), \beta(r_0))$, where $(\alpha(r_0), \beta(r_0))$ is the polarization determined by the vector $[u(r_0), v(r_0)]^T$. It then follows from Theorem 1 that

$$\begin{bmatrix} u(r_0) \\ v(r_0) \end{bmatrix} = \mathcal{P}(\alpha(r_0), \beta(r_0)).$$ (30)

The equalities in (29)–(30) lead to the following:

$$\begin{bmatrix} u(r_0) & v(r_0) \\ v(r_0) & -u(r_0) \end{bmatrix} \begin{bmatrix} s^{(E)}(r_0) \\ s^{(M)}(r_0) \end{bmatrix} = \mathcal{P}(\alpha(r_0), \beta(r_0)).$$

Due to the orthogonality between $[u(r_0), v(r_0)]^T$ and $[v(r_0), -u(r_0)]^T$, we obtain that $s^{(M)}(r_0) = 0$. It therefore holds that at any feasible solution, $s^{(M)}(r) = 0$, for which we can choose $w^{(M)} = 0$. This implies that if

$$w^{(E)}_{opt}, w^{(M)}_{opt}$$

is an optimal solution to problem (16), then the electric component $w^{(E)}_{opt}$ is a solution to the 1-D electric dipole array problem in (21). Thus

$$\tau^{opt} \leq \tau^{opt}(\alpha(r_0), \beta(r_0)) = \tau^{opt}(\alpha, \beta).$$ (31)

Combining (28) and (31) completes the proof. \square

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